MHD waves of stratified fluxtubes: on the kink mode cut-off

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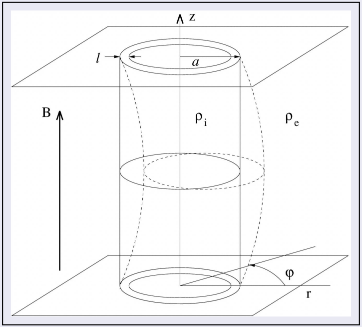
Magnetic tube oscillations (ideal MHD)

History

- 70’s - 80’s (Defouw, 1976)
- Motivated by magnetic field concentrations on the solar surface
- Inherently longitudinally stratified
- “Thin tube” approximations (Defouw, 1976; Roberts and Webb, 1978): “Sausage mode”
- Roberts and Webb (1979): non-stratified but “thick tube”
- Wilson (1979) (+ taut wire mode = kink mode)
- Spruit (1981) cut-off frequency for kink-mode
- Edwin and Roberts (1983) → Roberts et al. (1984) seminal paper on coronal seismology

Tube model

Edwin and Roberts (1983)



$$\exp(\imath(m\varphi - \omega t))$$

m : azimuthal wave numbers

ω : frequency

Straight tubes: homogeneous

Homogeneous in $z \Rightarrow \sim \exp(\imath(k_z z - \omega t))$

Iconic dispersion relation: Edwin and Roberts (1983)

$$\frac{I_m\left(\sqrt{-\mathcal{L}_\kappa^{\rm in}}R\right)}{\left(\sqrt{-\mathcal{L}_\kappa^{\rm in}}\right) I'_m\left(\sqrt{-\mathcal{L}_\kappa^{\rm in}}R\right)} B_{\rm in}\mathcal{L}_{\rm A}^{\rm in} - \frac{K_m\left(\sqrt{-\mathcal{L}_\kappa^{\rm ex}}R\right)}{\left(\sqrt{-\mathcal{L}_\kappa^{\rm ex}}\right) K'_m\left(\sqrt{-\mathcal{L}_\kappa^{\rm ex}}R\right)} B_{\rm ex}\mathcal{L}_{\rm A}^{\rm ex} = 0 \ . \tag{1}$$

with:

$$\mathcal{L}_\kappa = \frac{(\omega^2 - k_z^2 v_{\rm s}^2)(\omega^2 - {\omega_{\rm A}}^2)}{(v_{\rm s}^2 + v_{\rm A}^2)(\omega^2 - {\omega_{\rm c}}^2)} \tag{2}$$

$$B\mathcal{L}_{\rm A} = \rho(\omega^2 - {\omega_{\rm A}}^2) \tag{3}$$

Straight tubes: longitudinally stratified

Andries et al. (2005)

$\beta = 0$, $\vec{g} = 0$, longitudinal density variation $\rho(z)$
$\Longrightarrow$ Separable

$$\frac{\partial^2 p_{\rm T}}{\partial r^2} + \frac{1}{r}\frac{\partial p_{\rm T}}{\partial r} - \left(\frac{m^2}{r^2} - \frac{\mu}{B^2}\rho\mathcal{L}_{\rm A}\right) p_{\rm T} = 0.$$

with the Alfvén operator:

$$\rho\mathcal{L}_{\rm A} = \rho\omega^2 + \frac{B^2}{\mu}\frac{\partial^2}{\partial z^2} = \rho\left(\omega^2 + v_{\rm A}^2\frac{\partial^2}{\partial z^2}\right) .$$

Thin tubes again: $\beta = 0$, $m = 1$ kink mode

(Dymova and Ruderman, 2005)

$$\left((\rho_\mathrm{i} + \rho_\mathrm{e})\omega^2 + 2B^2 \frac{\partial^2}{\partial z^2} \right) \xi_r = 0$$

Verth and Erdélyi (2008), Ruderman et al. (2008): expanding field again

$$\left((\rho_\mathrm{i} + \rho_\mathrm{e})\omega^2 + 2B^2 \frac{\partial^2}{\partial z^2} \right) \frac{\xi_r}{R(z)} = 0$$

Relation with Andries et al. (2005)?

Replace longitudinal quantum numbers with longitudinal operators

$$\left(\frac{I_m\left(\sqrt{-B_{\rm in}^{-1}\mathcal{L}_{\rm A}^{\rm in}}R\right)}{\left(\sqrt{-B_{\rm in}^{-1}\mathcal{L}_{\rm A}^{\rm in}}\right) I'_m\left(\sqrt{-B_{\rm in}^{-1}\mathcal{L}_{\rm A}^{\rm in}}R\right)} B_{\rm in}\mathcal{L}_{\rm A}^{\rm in} - \frac{K_m\left(\sqrt{-B_{\rm ex}^{-1}\mathcal{L}_{\rm A}^{\rm ex}}R\right)}{\left(\sqrt{-B_{\rm ex}^{-1}\mathcal{L}_{\rm A}^{\rm ex}}\right) K'_m\left(\sqrt{-B_{\rm ex}^{-1}\mathcal{L}_{\rm A}^{\rm ex}}R\right)} B_{\rm ex}\mathcal{L}_{\rm A}^{\rm ex} \right) \xi_r(z) = 0 \,. \quad (4)$$

Challenge!

Generalize the above operator function solution to:

- Include tube expansion
- Include pressure effects: $\beta \neq 0$
- Include gravity: i.e. buoyancy

Assumptions

- Neglect curvature of field lines
- Neglect curvature of perpendicular plane

All satisfied as long as:

$$\frac{1}{B}\frac{\mathrm{d}B}{\mathrm{d}z} \ll \frac{1}{R}$$

Perpendicular invariance

Separation of variables:

$$\mathcal{L}_{\rm A} B^{\frac{1}{2}} \mathcal{L}_{\rm s} B \mathcal{L}_{\rm c}^{-1} B^{-\frac{3}{2}} Z(z) = -\lambda Z(z) \, , \quad (5)$$

$$\frac{1}{B} \nabla_{\perp}^2 P(\vec{r}_{\perp}) = \lambda P(\vec{r}_{\perp}) \, . \quad (6)$$

For expanding field:

$$\begin{aligned} \nabla \cdot \nabla_{\perp} &= \frac{1}{h_1 h_2 h_3} \left(\partial_1 \left(h_2 h_3 \frac{1}{h_1} \partial_1 \right) + \partial_2 \left(h_1 h_3 \frac{1}{h_2} \partial_2 \right) \right) \\ &= \frac{1}{h_1 h_2} \left(\partial_1 \left(\frac{h_2}{h_1} \partial_1 \right) + \partial_2 \left(\frac{h_1}{h_2} \partial_2 \right) \right) \quad (7) \end{aligned}$$

Diverging tube

$\psi \approx r(z)^2 B(z)/2$ with the normal and azimuthal scale factors $h_\psi \approx 1/r(z)B(z)$ and $h_\theta = r(z)$

Flux coordinates

$$\frac{\partial}{\partial\psi}\left(2\psi\frac{\partial p'_{\mathrm{T}}}{\partial\psi}\right) - \frac{1}{2\psi}m^2 p'_{\mathrm{T}} = \lambda p'_{\mathrm{T}} \; .$$

solved by $p'_{\mathrm{T}} = I_m(\sqrt{2\lambda\psi})$ or $p'_{\mathrm{T}} = K_m(\sqrt{2\lambda\psi})$.

Impedances again

Thus formally:

$$\frac{\sqrt{2\psi} I_m\left(\mathcal{L}_\kappa^{\rm in}\sqrt{2\psi}\right)}{\mathcal{L}_\kappa^{\rm in} I'_m\left(\mathcal{L}_\kappa^{\rm in}\sqrt{2\psi}\right)} h_\psi^{\rm in} B_{\rm in}\mathcal{L}_{\rm A}^{\rm in} - \frac{\sqrt{2\psi} K_m\left(\mathcal{L}_\kappa^{\rm ex}\sqrt{2\psi}\right)}{\mathcal{L}_\kappa^{\rm ex} K'_m\left(\mathcal{L}_\kappa^{\rm ex}\sqrt{2\psi}\right)} h_\psi^{\rm ex} B_{\rm ex}\mathcal{L}_{\rm A}^{\rm ex} = 0\ . \quad (8)$$

"thin flux tubes" once more

Small argument expansions in Bessel functions $\Rightarrow$

$$\left(B_{\rm in}(z)\mathcal{L}_{\rm A}^{\rm in} + B_{\rm ex}(z)\mathcal{L}_{\rm A}^{\rm ex}\right)\xi_r(z) = 0\ . \quad (9)$$

- Also valid for $m > 1$!
- It is about time to discuss $\mathcal{L}_{\rm A}$ in more detail!

$\mathcal{L}_\mathrm{A}$

$$\left[\left(\vec{B}\cdot\nabla\right)\vec{b_\perp}+\left(\vec{b_\perp}\cdot\nabla\right)\vec{B}\right]_1 = \left(\vec{B}\cdot\nabla\right)b_1+\frac{b_1 B}{h_1 h_3}\frac{\partial h_1}{\partial x_3}=\frac{1}{h_1}\left(\vec{B}\cdot\nabla\right)h_1 b_1 \quad (10)$$

$$= \frac{1}{h_1}\left(\vec{B}\cdot\nabla\right)h_1^2\left(\vec{B}\cdot\nabla\right)\frac{\xi_1}{h_1}\,, \quad (11)$$

$$\left[\left(\vec{B}\cdot\nabla\right)\vec{b_\perp}+\left(\vec{b_\perp}\cdot\nabla\right)\vec{B}\right]_2 = \left(\vec{B}\cdot\nabla\right)b_2+\frac{b_2 B}{h_2 h_3}\frac{\partial h_2}{\partial x_3}=\frac{1}{h_2}\left(\vec{B}\cdot\nabla\right)h_2 b_2 \quad (12)$$

$$= \frac{1}{h_2}\left(\vec{B}\cdot\nabla\right)h_2^2\left(\vec{B}\cdot\nabla\right)\frac{\xi_2}{h_2}\,. \quad (13)$$

By flux conservation $r^2(z)B(z)=C \Rightarrow h_\psi \sim h_\theta$ ($h_\psi \approx 1/r(z)B(z)$ and $h_\theta = r(z)$.)

$$\mathcal{L}_\mathrm{A} = r(z)B(z)\partial_\parallel^2\frac{1}{r(z)} - \frac{\rho(z)}{B(z)}\frac{\partial^2}{\partial t^2}\,. \quad (14)$$

"thin flux tubes" for the very last time

Immediately recover Dymova and Ruderman (2006), Ruderman et al. (2008)

$$\left((\rho_\mathrm{i} + \rho_\mathrm{e})\omega^2 + 2B^2 \frac{\partial^2}{\partial z^2}\right) \frac{\xi_r}{R(z)} = 0$$

Also valid for $m > 1$!

BUT How to reconcile with:

Spruit (1981)

$$\left((\rho_\mathrm{i} + \rho_\mathrm{e})\omega^2 + g(\rho_\mathrm{e} - \rho_\mathrm{i})\frac{\partial}{\partial z} + B^2 \frac{\partial^2}{\partial z^2}\right) \xi_r = 0 \ , \qquad (15)$$

or actually

$$\left((\rho_\mathrm{i} + \rho_\mathrm{e})\omega^2 + B\left(\frac{\partial B}{\partial z}\right)\frac{\partial}{\partial z} + B^2 \frac{\partial^2}{\partial z^2}\right) \xi_r = 0? \qquad (16)$$

"thin flux tubes" for the very last time (continued)

Now find:

$$\mathcal{L}_{\rm A} = \left(\partial_\parallel + \frac{1}{h_i}(\partial_\parallel h_i)\right) B \left(\partial_\parallel - \frac{1}{h_i}(\partial_\parallel h_i)\right) - \frac{\rho}{B}\frac{\partial^2}{\partial t^2} , \quad (17)$$

$$= \partial_\parallel B \partial_\parallel - \left(\partial_\parallel \frac{B}{h_i}(\partial_\parallel h_i) + \frac{B}{h_i^2}(\partial_\parallel h_i)^2\right) - \frac{\rho}{B}\frac{\partial^2}{\partial t^2} . \quad (18)$$

Hence Spruit needs additional approximation: longitudinal variation of perturbation is faster than that of equilibrium!

On the cut-off frequency

Spruit (1981)

Remove first order derivative by "integrating factor" $\sqrt{B}$)

$$\left((\rho_{\rm i} + \rho_{\rm e})\omega^2 + B^2 \frac{\partial^2}{\partial z^2} - B^2 \left(\left(\frac{1}{4H} \right)^2 - \frac{\partial}{\partial z} \left(\frac{1}{4H} \right) \right) \right) \sqrt{B}\xi_r = 0 \quad (19)$$

with:

$$\frac{1}{4H} = \frac{1}{\sqrt{B}} \frac{\partial \sqrt{B}}{\partial z} \quad (20)$$

Klein-Gordon (if isothermal and constant β).
Cut-off frequency is of same order as the terms neglected in $\mathcal{L}_{\rm A}$.
More general treatment clarifies the terms making up the cut off are absent
All kink-modes may propagate upwards!

Summary

- Generalization of tube dispersion relation for stratified tubes
- Valid for 'slowly diverging' tubes although they are 'thick'
- Recover limiting cases for 'thin tubes'
- In particular the kink mode for which there is NO cut-off for the isothermal case

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