Introduction to Digital Signal Processing

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Contents

- Linear Systems
  - Linear 2 port
  - RC filter
  - Time response
  - Poles and zeros
  - Transforms
- Sampled Data
  - Z transform
- Digital Signal processing
  - Averaging
  - Discrete time convolution
  - Signal Flow Diagrams
  - Recursion and Filters
- DSP in practice

1 Linear Systems

- Where a mathematically linear relationship exists between input and output:
  - Ideal amplifier
  - Ideal transducer
  - Radio astronomy receiver!
- Note a detector e.g. a square law or power detector $V_{out} \propto V_{in}^2$ is distinctly non-linear
- The Transfer function *H(p)* relates input to output for a linear two-port

1.1 Linear Two-port Transfer function

$V_i(p)$ $V_o(p)$

$H(p)$
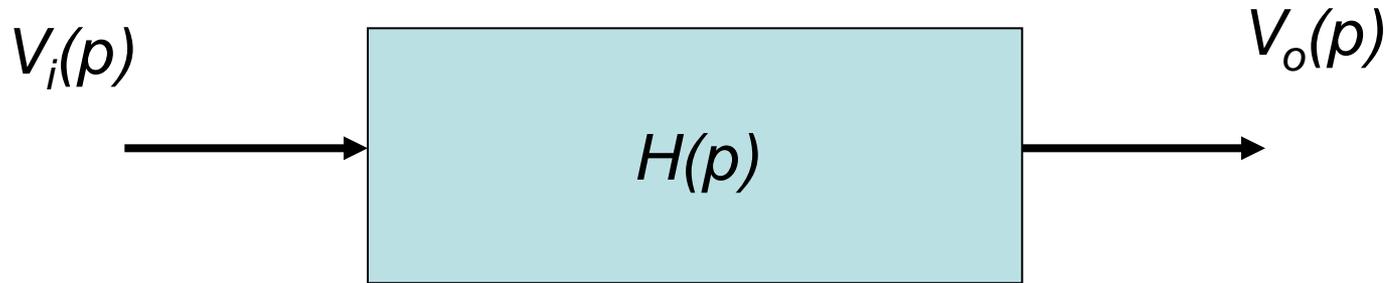
$$H(p) = \frac{V_o(p)}{V_i(p)}$$

Note if p = jω then *H(jω)* gives the frequency response:
real and imaginary parts --- amplitude and phase response.

p notation (s in some books) comes from assuming solution to a differential
equation describing the system is of the form $V_o(p) = V_o \ e^{pt}$

Example: RC circuit
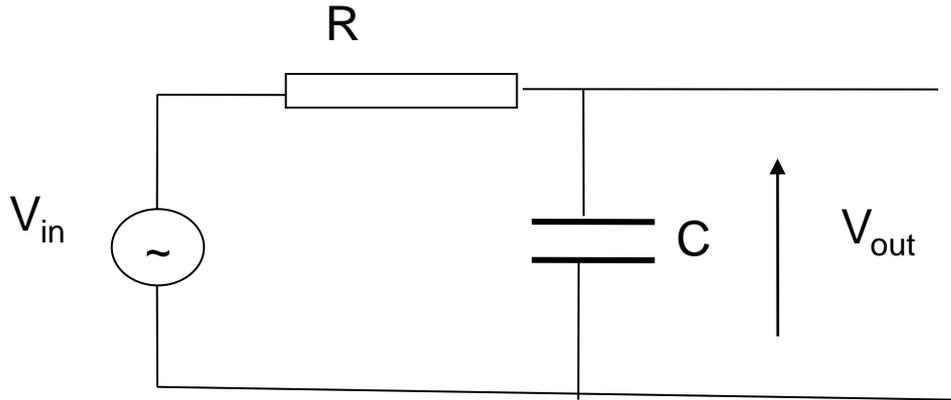
$$C \frac{dV_{out}}{dt} + \frac{V_{out}}{R} = \frac{V_{in}}{R}$$

Try a driving function

$$V_{in} = V_i e^{pt}$$

with solution of the form

$$V_{out} = V_{out} e^{pt}$$

gives

$$\frac{V_o}{V_i} = \frac{1}{1 + pRC} = H(p)$$

1.2 Time response

- Step response H(t) Heaviside's Step function = 1 (t>0), = ½ (t=0), =0 (t<0)
  - useful for describing causal signals

- Impulse response δ(t)=0 t≠0, area=1

- Related by the integral

t

t

$$\int_{-\infty}^{+\infty} \delta(t')dt' = 1 \text{ if t>0 and } = 0 \text{ if t<0}$$

$$= \text{H}(t)$$

so we can write

NB H(t) ≠ *H(p)*

$$\delta(t) = \frac{d\text{H}(t)}{dt}$$
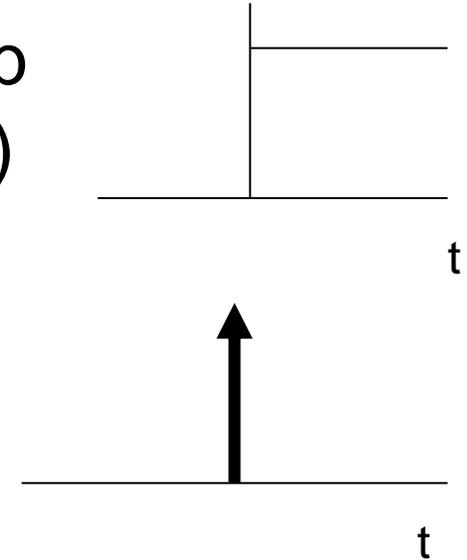
For the RC circuit

Step response

$$v_0(t) = (1 - e^{\frac{-t}{RC}})\mathrm{H}(t)$$
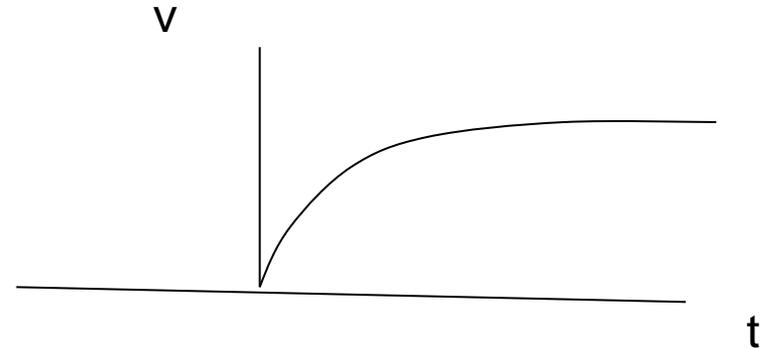
Impulse response

$$h(t) = \frac{dv_o}{dt} = \frac{1}{RC}e^{\frac{-t}{RC}}\mathrm{H}(t) + (1 - e^{\frac{-t}{RC}})\delta(t)$$

$$= \frac{1}{RC}e^{\frac{-t}{RC}} \quad \text{for } t > 0, \quad = 0 \text{ for } t < 0$$
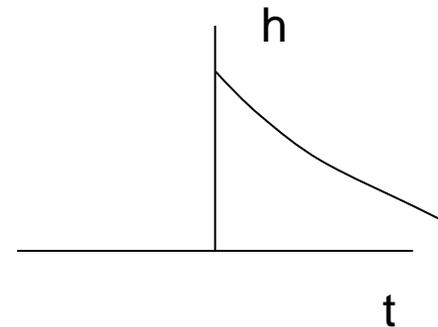
1.3 Poles and Zeros

E.g. for the RC circuit: $\dfrac{V_o}{V_i} = \dfrac{1}{1 + pRC}$ has a pole at $p = -1/RC$ and a zero at infinity

A general transfer function

$$H(p) = \frac{a_m p^m + a_{m-1} p^{m-1} + \ldots\ldots + a_1 p^1 + a_0}{b_n p^n + b_{n-1} p^{n-1} + \ldots\ldots + b_1 p^1 + b_0}$$

$$= k \frac{(p - z_1)(p - z_2)\ldots\ldots(p - z_n)}{(p - q_1)(p - q_2)\ldots\ldots(p - q_m)}$$

has m poles and n zeros. The coefficients a, b are
always real so the poles and zeros are either on
the real axis or occur in complex pairs on an
Argand diagram (Hermitian property)

Imag

-1/RC

Real

P plane
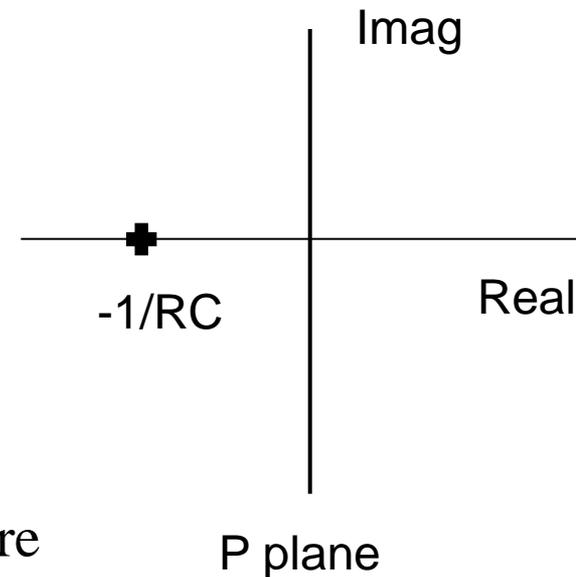
1.4 Transforms

Fourier transform relates time and frequency domains (e.g. Bracewell)

$$G(f) = \int\limits_{-\infty}^{\infty} g(t)e^{-i2\pi ft}dt$$

with the inverse

$$g(t) = \int\limits_{-\infty}^{\infty} G(f)e^{+j2\pi ft}df$$

Transforms (2) :

- Laplace transform used in signal analysis since causal (no signals for t<0)

$$F(p) = \int\limits_{o}^{\infty} f(t)e^{-pt}dt$$

Inverse:

$$f(t) = \frac{1}{2\pi j} \int\limits_{\sigma - j\infty}^{\sigma + j\infty} F(p)e^{pt}dp$$

- complex integration needing calculus of residues

actually not so useful.

Both transforms relate the time and frequency response, though the Laplace transform is more directly related to the transfer function

2. Sampled Data

- ADC usually converts an incoming signal to digital values at a constant rate – the sampling rate (e.g. 4 Gsamples/sec in e-MERLIN EVLA and ALMA per channel)
- Number of bits important (1, 2, 3, 6, 8, 14)
- A wide variety of types available – SKA needs few (4??) bits but high speed (several Gs/s) and low power
- Sampled at a rate >2 highest frequency in the signal – otherwise loss of information – Shannon sampling theorem.
- Nyquist rate = 2 x bandwidth of signal

- Suppose we have a set of values of a sampled voltage [v]:
  v_0 , v_1, v_2 , v_3 , v_4 , etc. sampled at a rate 1/T, T=the sampling period

- This forms a time series   . . . . A sequence of values

2.1Time Series and the Z transform

The times series can be written:

$$v = v_0 \delta(t) + v_1 \delta(t-T) + v_2 \delta(t-2T) + v_3 \delta(t-3T) + ...$$

The Laplace transform of this is

$$V(p) = v_0 + v_1 e^{-pT} + v_2 e^{-2pT} + v_3 e^{-3pT} + ...$$

Note the Fourier transform is

$$V(\omega) = v_0 + v_1 e^{-j\omega T} + v_2 e^{-2j\omega T} + v_3 e^{-3j\omega T} + ...$$

Introduce a new variable $z = e^{pT}$ then we have

$$V(z) = v_0 + v_1 z^{-1} + v_2 z^{-2} + v_3 z^{-3} + ...$$

This is the z transform of the time series

2.2 Shift operators

Supposed p has a real and imaginary parts:

$p = \sigma + j\omega$ then $z = e^{pT} = e^{\sigma T} e^{j\omega T}$

$e^{-j\omega T}$ represents a phase shift corresponding to a time delay of T

$e^{+j\omega T}$ represents a time advance of T

So z is a shift operator, advancing by T

and z^{-1} is a backshift operator.

In summary the z transform is a sequence of samples [v] is

$$V(z) = \sum_{k} v_k z^{-k}$$

If $z = e^{j\omega T}$ then $V(z)$ gives the frequency spectrum of the sampled voltage

Note – we have poles and zeros on the z-plane, just like the p plane

3. Digital Signal Processing

- Fourier transform (useful for continuous signals) reduces to the z transform when we have discrete data – we will see later how this can be used

- Obvious forms of analysis – averaging of the data – gives data reduction

- Two main types:
  - Boxcar averaging
  - Running average

3.1 Boxcar Averaging

n=0

n=1

n=2

$x_1 \ldots x_m$ | $x_{m+1} \ldots x_{m+m}$ | $x_{2m+1} \ldots \ldots x_{3m}$
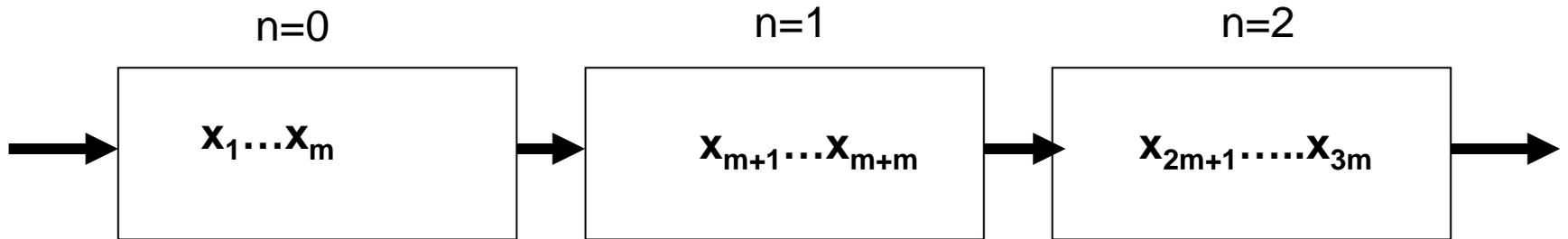
- Each boxcar contains m data points
- Gives an average y_n for the nth box
- For a series of values [x] (instead of v now – since could be anything)

$$y_n = \sum_{k=1}^{m} x_{mn+k}$$

3.2 Running (moving) average

E.g. a 4 point running average:

$$y_k = \frac{1}{4}\left(x_k + x_{k-1} + x_{k-2} + x_{k-3} \right)$$

- An example of a digital filter
- Impulse response:
  - Input is $h_k = 1$ for $k=0$, $h_k=0$ for $k \neq 0$
  - Output becomes:

$y_0 = 0.25(1+0+0+0) = 0.25$

$y_1 = 0.25(0+1+0+0) = 0.25$

$y_2 = 0.25$

$y_3 = 0.25$

$y_4 = 0$ etc.

h_k

k
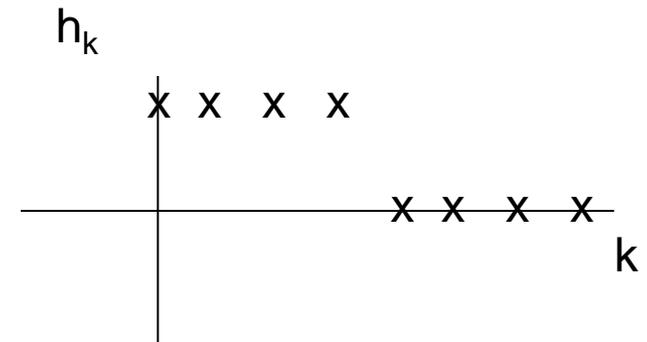
Impulse response of a moving average filter

Running average (2)

The Z transform for this filter is by inspection

$$H(z) = \frac{1}{4}\left(1 + z^{-1} + z^{-2} + z^{-3}\right)$$ and this is the

Transfer Function of such a filter with a frequency response

$$H(j\omega) = \sum_{k=-\infty}^{\infty} h_k e^{-j\omega Tk}$$ with Gain G and phase θ:

$$G(\omega) = \left| H(j\omega) \right| \quad \text{and} \quad \theta(\omega) = \arg(H(j\omega))$$

3.3 Signal Flow Diagram
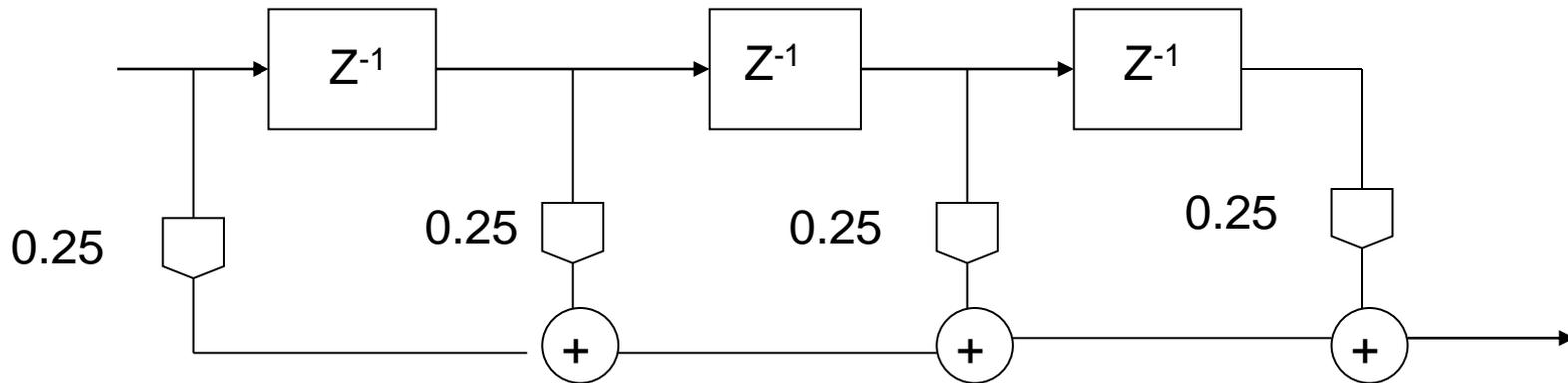
- Each z^{-1} block is a back shift i.e. a delay of one sample period (T) – a shift register
- Known as a Transversal Filter
- If a finite number of elements – gives a Finite Impulse Response (FIR) Filter

3.4 Recursion and filters

Suppose for exmple we have a response where

$$y_k = x_k - x_{k-1} + 0.3 y_{k-1}$$

this is recursive, and produces an Auto Regressive time series

Note this is a difference equation.

The impulse response is infinite (asymptotic to 0)
 and produces an Infinite Impulse Response (IIR) filter.

---FIR filters can be recursive or non-recursive

---IIR filters are always recursive, use relatively few poles
   but need high accuracy (bits)

In general we can write:

$$y_k = \sum_1^m a_i x_{k-i} + \sum_1^n b_i y_{k-i}$$

which is a digital filter, but also represents an auto regressive
 moving average (ARMA) time series :::

3.5 General transfer function:

Transfer function for the general digital filter is:

$$H(z) = \frac{\sum_{i=0}^{m} a_i z^{-i}}{1 - \sum_{l=1}^{n} b_l z^{-l}}$$ with a frequency response

$$H(j\omega) = \frac{\sum_{i-0}^{m} a_i e^{-j\omega Ti}}{1 - \sum_{l=1}^{n} b_l e^{-j\omega Tl}}$$

4.DSP in Practice

- FIR filters in common use– see handout for an example. Can have large number of elements – 150 common

- Simple averaging commonly used in simple processing

- Need for spectral data:
  - Use Direct Fourier Transform (DFT)

  $$G(f) = \sum_{1}^{n} g_k (\cos(2\pi f k T) + j \sin(2\pi f k T)$$

  - Fast Fourier transforms used for large (>32) numbers of elements (see refs) when speed is a premium

4.1 DSP uses:

- Data windows in measuring spectra – Hanning, Hamming, and Tapered rectangular windows (handout)
- Decimation:
    - Reducing data rate per channel
    - Makes use of parallel structures in hardware
    - Demultiplexing
- Decimation plus FIR filters and FFT --- polyphase filters
- Delay, Phase rotation, frequency shifting, correlation
- Formatting, packetisation
- In practice use design tools: Simulink, Matlab, etc.
- Can be implemented on specialist DSP chips or FPGAs

4.2 Some examples

Motorola evaluation board
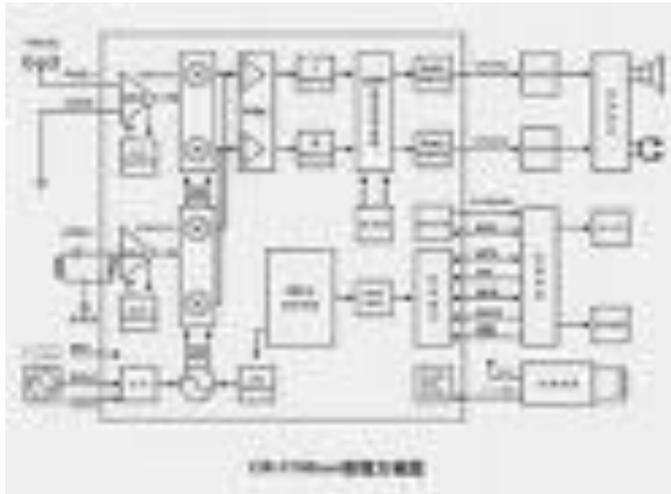
Chip used in a nome cinema sound system

Xilinx Spartan FPGA

CT1100 digital radio


NEC audio DSP chip

High Definition Radio
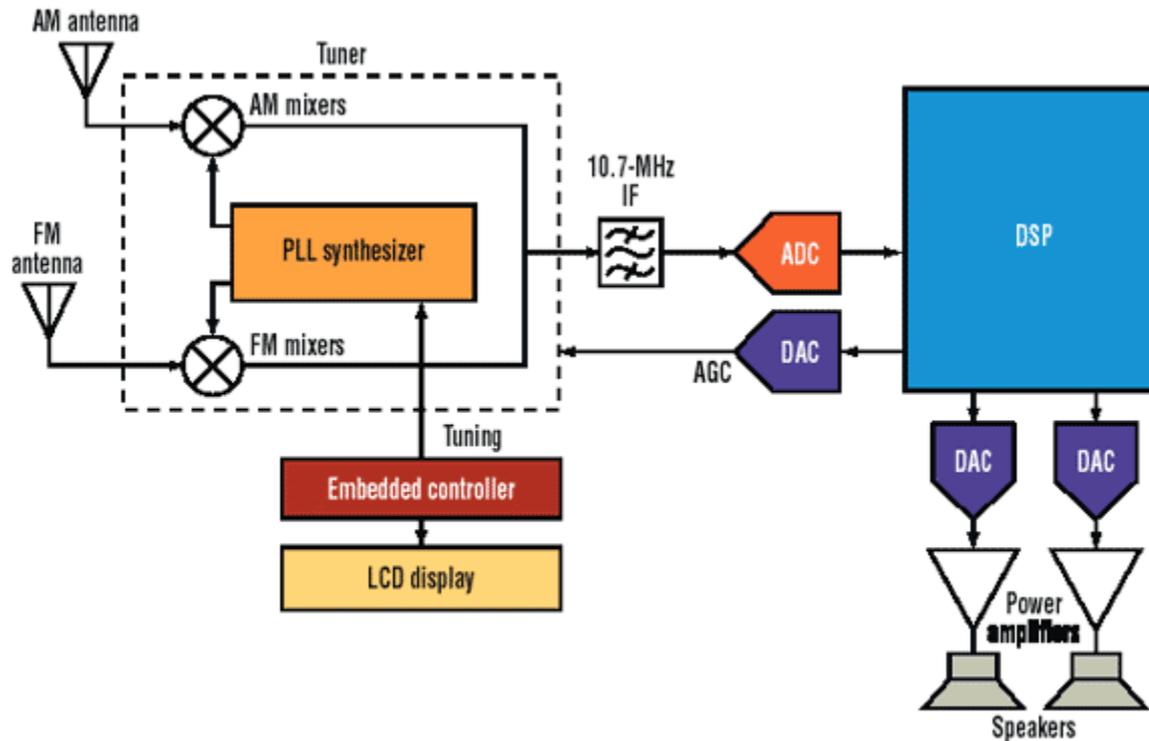
3. In a typical HD Radio receiver, the partitioning of the blocks is a function of the chips.

Uses low power Texas DSP chip (Electronics Design March 2006)
Chip does decompression, de-interleaving, decoding and demodulation

5. Bibliography

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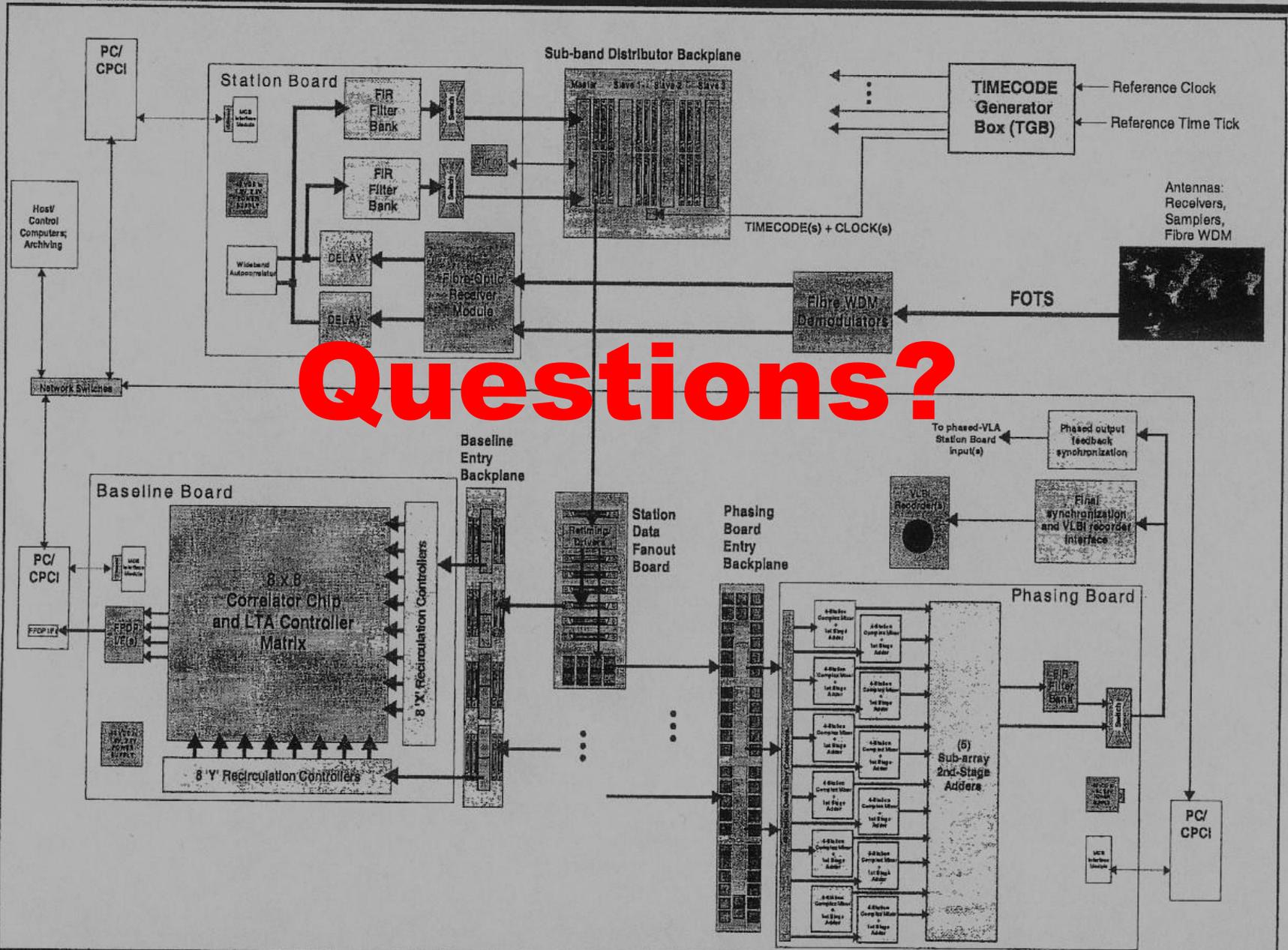
Problems for the Student:

1. Write down the first five terms of the transfer function for a FIR filter having an impulse response *h(t)=exp(-t/T)*, where *T* is the sampling interval. Sketch a signal flow diagram for the filter

2. Write down the excursion relation (a diffrence relation) for the impulse response in Q1 and hence the transfer function for an IIR filter

3. A random signal of variance σ^2 is averaged by a) a boxcar integrator of length 5 b) a FIR filter as in Q1. Calculate the output standard deviation.