Workshop 9: the microwave background radiation

1. Times

Calculate the times, t, after the Big Bang at which equipartition, recombination, decoupling, and the present occur in an open universe with Hubble constant $H_0 = 70\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$, cold (dark plus baryonic) matter density parameter $\Omega_{m0} = 0.2$, baryon density parameter $\Omega_{b0} = 0.04$, and zero cosmological constant using the cosmological solution for redshift

$$(1+z)^{-1} = \frac{\Omega_{m0}}{2\,(1-\Omega_{m0})}\,(\cosh\psi - 1)$$
$$\sinh\psi - \psi = 2\frac{H_0 t}{\Omega_{m0}}\,(1-\Omega_{m0})^{3/2}$$

where ψ is the development angle, given that the redshifts of equipartition, recombination, and decoupling are approximately

$$z_{\mathrm{eq}} = 2.5\times 10^4\,\Omega_{m0}\,h_{100}^2$$
$$z_{\mathrm{rec}} = 1.4\times 10^3\,\left(\Omega_{b0}h_{100}^2\right)^{0.02}$$
$$z_{\mathrm{dec}} = 1.1\times 10^3\,\left(\frac{\Omega_{m0}}{\Omega_{b0}}\right)^{0.02} \quad .$$

Which of these redshifts/times corresponds to the radiation field shown by CMB measurements? What fraction of the history of the Universe is concealed from us by photon opacity?

2. Time of equipartition

A better calculation of the time of equipartition takes account of the radiation-dominated nature of the Universe at $t < t_{\mathrm{eq}}$. The energy density was then dominated by the redshifted microwave background radiation. Today that radiation has energy density

$$u_{\gamma 0} = a_R T_{r0}^4$$

with a_R being the radiation constant and $T_{r0} = 2.728$ K being the radiation temperature of the microwave background radiation. Use the expression for the scale factor, $a(t)$, in a radiation-dominated Universe

$$a = a_{\mathrm{eq}}\left(\frac{32\,\pi\,G\,\rho_{\mathrm{eq}}\,t^2}{3}\right)^{1/4}$$

to calculate a revised time for equipartition in a Universe with $h_{100} = 0.7$, $\Omega_{\mathrm{m}0} = 0.2$, and $\Omega_{\mathrm{b}0} = 0.04$, as in question 1. Comment on your result. You will need to know that the temperature of the microwave background radiation scales with redshift as $1+z$.

3. Recombination

(a) Taking the redshift of recombination from question 1, calculate the temperature of the microwave background at that time if the temperature of the cosmic microwave background today is 2.728 K.

(b) Recombination is usually calculated at the time when the matter in the Universe is 50% ionised. Taking that to be the case, compare the energy of typical photons in the Universe at recombination with the ionisation energy of hydrogen (13.6 eV) and comment on the result.

(c) The density of matter in the Universe and the density of radiation in the Universe scale as $(1+z)^3$ and $(1+z)^4$ respectively, during epochs at which matter and radiation interact weakly. If matter and radiation had equal density at the epoch of equipartition, what is the density of matter today?

4. The Sunyaev-Zel'dovich effect

Estimate the size of the central Sunyaev-Zel'dovich effect from a cluster atmosphere modelled as a uniform sphere of gas with radius 1 Mpc, density 10^{-25} kg m^{-3}, and temperature 6×10^7 K, expressed as (a) central y parameter (proportional to the central surface brightness change in the microwave background radiation) and (b) total Y parameter (proportional to the flux density of the cluster), if the cluster is at redshift $z = 0.1$ (where the angular size corresponding to 1 Mpc is 9.0 arcmin) or at $z = 1$ (where the angular size is 2.1 arcmin).

y and Y are defined by

$$y = \int dl\, \sigma_T n_e \left(\frac{k_B T_e}{m_e c^2}\right)$$

$$Y = \int d\Omega\, y$$

where dl is an element of length along the line of sight and $d\Omega$ is an element of solid angle about the line of sight.

5. Clusters of galaxies

Estimate the mass of the cluster in question 4 assuming that the gas in the cluster is in hydrostatic equilibrium, so that the gravitational potential energy of a proton is approximately the same as its kinetic energy. Compare this with the gas mass of the cluster and comment on your result.

6. Kinematic SZ effect. The kinematic SZ effect causes a fractional change in brightness of the CMB of order $\tau_e\, v_r/c$ where v_r is the line-of-sight speed of the cluster relative to the general expansion of the Universe. Estimate the velocity of a cluster with gas temperature 6×10^7 K that would be required for that cluster's kinematic SZ effect to be 10% the size of the thermal effect (expressed in units of y).

Useful constants

gravitational constant, $G = 6.7 \times 10^{-11}\ \mathrm{kg^{-1}\,m^3\,s^{-2}}$
velocity of light, $c = 3.0 \times 10^{8}\ \mathrm{m\,s^{-1}}$
electron mass, $m_e = 9.1 \times 10^{-31}\ \mathrm{kg}$
proton mass, $m_p = 1.7 \times 10^{-27}\ \mathrm{kg}$
Boltzmann constant, $k_B = 1.3 \times 10^{-23}\ \mathrm{J\,K^{-1}}$
Thomson scattering cross-section, $\sigma_T = 6.7 \times 10^{-29}\ \mathrm{m^2}$
radiation constant, $a_\mathrm{R} = 7.6 \times 10^{-16}\ \mathrm{J\,m^{-3}\,K^{-4}}$
Hubble constant, $H_0 = 100\, h_{100}\ \mathrm{km\,s^{-1}\,Mpc^{-1}}$
1 AU $= 1.5 \times 10^{11}\ \mathrm{m}$
1 pc $= 3.1 \times 10^{16}\ \mathrm{m}$
1 arcsec $= 4.8 \times 10^{-6}\ \mathrm{rad}$