

2. Equations of Stellar Structure

We already discussed that the structure of stars is basically governed by three simple laws, namely hydrostatic equilibrium, energy transport and energy generation. In this section we will express these laws in precise mathematical terms.

Throughout this course, we will consider only a spherically symmetric star. Since most stars evolve very slowly, we will regard it as a non-changing object over a short period of time. For such a star, all the physical quantities are just a function of radius, r . At any given radius, all the microscopic quantities (such as entropy) depend on the density, temperature and chemical composition

$$\rho, T, X_i \tag{1}$$

The stellar structure is determined by four basic equations, supplemented by the equation of state, the law of opacity, and boundary conditions.

2.1. Mass conservation

Let $M(r)$ be the mass contained within radius r , then the mass contained with the shell from r to $r + dr$ is then

$$M(r + dr) - M(r) = \rho(r)dV, \tag{2}$$

where $dV = 4\pi r^2 dr$ is the volume of the shell. Now from the definition of derivative, the left-hand-side can be rewritten as,

$$\frac{dM(r)}{dr} dr = \rho(r)4\pi r^2 dr \tag{3}$$

Cancelling out the factor dr , we obtain

$$\frac{dM_r}{dr} = \rho(r)4\pi r^2 \tag{4}$$

Hereafter, we will, following the conventions in most astronomy textbooks, write the mass within radius r as M_r .

2.2. Hydrostatic equilibrium

For a star in hydrostatic equilibrium, gravity is balanced by pressure. Let us again consider a small element in a shell from r to $r + dr$. The area of this element is dA .

Let us establish a coordinate system where the radial coordinate increases outwards. Then the pressure force acting on the inner side of radius r is positive, while the pressure force acting on the outer side of radius $r + dr$ is negative.

The total pressure force is then

$$P(r)dA - P(r + dr)dA = [P(r) - P(r + dr)] dA = -\frac{dP}{dr}drdA. \quad (5)$$

The gravity force, due to spherical symmetry, points toward the center and therefore has a negative sign

$$-\frac{GM_r dm}{r^2}, \quad (6)$$

where dm is the mass of the element. Obviously

$$dm = \rho(r)dV = \rho(r)dAdr \quad (7)$$

Therefore the gravity is given by

$$-\frac{GM_r \rho(r)dAdr}{r^2} \quad (8)$$

So from Newton's second law, we have

$$\rho(r)dAdr \frac{d^2r}{dt^2} = -\frac{dP}{dr}drdA - \frac{GM_r \rho(r)dAdr}{r^2} \quad (9)$$

We can cancel out the factors dr and dA , and arrive at:

$$\rho(r) \frac{d^2r}{dt^2} = -\frac{dP}{dr} - \frac{GM_r \rho(r)}{r^2} \quad (10)$$

If the star is in hydrostatic equilibrium, then the sum of all forces must vanish, i.e.,

$$-\frac{dP}{dr} - \frac{GM_r \rho(r)}{r^2} = 0 \quad (11)$$

Moving the second term to the right-hand-side, we have

$$\frac{dP(r)}{dr} = -\frac{GM_r \rho(r)}{r^2} \quad (12)$$

Since the right hand side is always negative, it means that the pressure gradient must be negative, in other words, the pressure decreases from the inner central region to the outer region, as one would expect.

2.3. Energy conservation

Stars lose energy via radiation. This radiation loss must be balanced by energy generated by nuclear reactions. The energy conservation equation expresses this in mathematical terms.

Let $L(r)$ be the energy flow across the sphere with radius r , in units of W, then the net energy loss in the shell from r to $r + dr$ is

$$L(r + dr) - L(r) = \frac{dL(r)}{dr}dr. \quad (13)$$

If ϵ is the energy generation rate per kg, then the total energy generated in the shell is

$$\epsilon dm = \epsilon \rho(r)4\pi r^2 dr \quad (14)$$

For the gas to be in thermal equilibrium, the radiation loss must be equal to the energy gain from nuclear burning. Therefore we have

$$\frac{dL(r)}{dr} dr = \epsilon \rho(r) 4\pi r^2 dr. \quad (15)$$

dr cancels out, so we have

$$\frac{dL_r}{dr} = \epsilon \rho(r) 4\pi r^2, \quad (16)$$

where, hence-after, we will write the energy flow across r as L_r .

- Since the right hand side must be positive, therefore L_r must be positive, so the energy flow rate L_r increases outwards.
- The energy generation rate depends on ρ, T, X_i , we will discuss this in more detail in §5.

2.4. Energy transport

We know that the Sun is hotter at the center, and cooler at the surface, so heat (energy) naturally flows from the centre to the surface. The energy transport equation describes how this energy transfer occurs.

There are three possibilities of energy transfer: conduction, radiation and convection.

There is no principle difference between conduction and radiation. In both cases, energetic particles collide with less energetic ones resulting in an exchange of energy.

In conduction, energy is carried by electrons while in radiation, energy is carried by photons. In most stars, we will learn, radiative transfer is much more important than conduction by electrons.

In both conduction and radiation, the flux of energy flow, in units of Joule per square meters per second, is directly proportional to the temperature gradient

$$f(r) = -\lambda \frac{dT(r)}{dr}, \quad (17)$$

where λ is the coefficient of conductivity.

- The negative sign ensures that the heat is always transferred from high temperature regions to low-temperature regions.
- The coefficient of conductivity λ depends on the T, ρ and chemical composition X_i , and we will derive it in §4.

Obviously the energy flow across a sphere of radius r is related to the flux by

$$L_r = 4\pi r^2 f(r) \quad (18)$$

Combining eqs. (17) and (18), we obtain

$$\frac{dL_r}{dr} = -\frac{1}{4\pi r^2 \lambda} L_r \quad (19)$$

Now the third mode of heat transfer is convection. Convection occurs when the temperature gradient is high. In this case, whole hot parcels of gas can travel over large distance without achieving thermal equilibrium with surrounding cold gas.

Due to this large travelling path (mean free path), convection is a global phenomenon, and is still ill understood. Convection occurs in many stars, including the Sun. This ill-understood convection is by far is the largest uncertainty in the theory of stellar structure and evolution. Simple semi-empirical laws (so-called mixing length theory) have been devised to describe the convection. We will return to this issue in §4.

Summary

So far we have derived four differential equations:

$$\frac{dM_r}{dr} = 4\pi r^2 \rho \tag{20}$$

$$\frac{dP}{dr} = -\frac{GM_r}{r^2} \rho \tag{21}$$

$$\frac{dL_r}{dr} = 4\pi r^2 \epsilon \rho \tag{22}$$

$$\frac{dT}{dr} = -\frac{1}{4\pi r^2 \lambda} L_r \tag{23}$$

There are four primary variables M_r, P, L_r, T in these equations, all as a function of radius r . We have three auxiliary equations

- ρ : equation of state, $P = P(\rho, T, X_i)$
- λ : coefficient of conductivity, $\lambda(\rho, T, X_i)$
- ϵ : nuclear fusion rate, $\epsilon(\rho, T, X_i)$

These are three key pieces of physics and we will discuss them in greater detail in §3-5.

So far, we have discussed the equations using r as the independent variable. Unfortunately it is inconvenient to specify the boundary conditions in r , as the outer radius R is not known before we solve the equations¹.

Fortunately, we frequently want to solve the stellar equations as a function of the mass M . As r and M_r have one-to-one correspondence, so we can use M_r as a more convenient independent variable. The four stellar equations can be rewritten as

$$\frac{dr}{dM_r} = \frac{1}{4\pi r^2 \rho} \tag{24}$$

$$\frac{dP}{dM_r} = -\frac{GM_r}{4\pi r^4} \tag{25}$$

¹Imagine the difficulty of trying to score in football when you have a moving goal post

$$\frac{dL_r}{dM_r} = \epsilon \tag{26}$$

$$\frac{dT}{dM_r} = -\frac{1}{16\pi^2 r^4 \lambda \rho} L_r \tag{27}$$

2.5. Boundary conditions

We have four differential equations for four primary variables, r , P , L_r and T , all as a function of the new independent variable M_r . On the right-hand-side, in addition to the four primary variables, we also have ρ , λ and ϵ . However, all three can be expressed as a function of (P, T, X_i) . For example, we can invert the equation of state $P = P(\rho, T, X_i)$ and express the density as $\rho = \rho(P, T, X_i)$. So the right-hand-side is also only related to the independent variable M_r and the four primary variables. With four differential equations for four variables, we can solve them with proper boundary conditions.

Two boundary conditions are obvious at the center. There can be no heat transfer across and no mass contained within a sphere of zero radius. Therefore we have

$$r = 0, \quad L_r = 0, \quad \text{for } M_r = 0 \tag{28}$$

We can also specify two boundary conditions at the surface. The simplest guess would be that the pressure and temperature are equal to zero as they are much smaller compared with the central values:

$$P = 0, \quad T = 0, \quad \text{for } M_r = M, \tag{29}$$

However, these naive guesses are not very good. The reason is that the “surface” of star is not well defined. A more rigorous treatment must use a more detailed model of the stellar atmosphere. We will discuss a simple stellar atmosphere model in Chapter 4 when we study the energy transport in stars.

- Notice that the boundary conditions are specified at two points, at the centre ($M_r = 0$) and at the surface ($M_r = M$). And furthermore, both the stellar luminosity L and R need to be found from solving the differential equations!
- Are the solutions unique to the differential equations? The so-called Russell-Vogt theorem claims the answer is yes, but in fact, in some cases, there could be multiple solutions. Fortunately for main sequence stars, the solution is unique over a wide range of stellar mass. So we will not be concerned with this fine point.

2.6. Complications

So far we have considered only a spherical star in hydrostatic equilibrium, there are many simplifications made

- Stars rotate, and this causes flattening, which breaks the spherical symmetry. Rotation can also cause mixing between the atmosphere, just like earth rotation can cause mixing of atmospheres in different regions.

- Stars do change with time, and in fact, as nuclear reaction proceeds, the element abundances change. So rigorously, one has to consider changes in the chemical composition X_i . This will one additional equation for each chemical element.
- We have briefly mentioned convection. This important energy generation mechanism is still ill understood. Convection can not only carry heat, it can also mix elements from very different parts of the stars (so-called dredge-ups), and cause more complications.

Exercise 2.1 *The period of revolution of the Sun around its axis is about 27 days. Estimate the ratio of the centrifugal acceleration and the gravitational acceleration at the equator. (answer: $\sim 1.9 \times 10^{-5}$)*

2.7. Some Applications of Stellar Equations

In this section, we shall derive the important virial theorem and present some examples of how to do orders of magnitude estimate. Through these, we will justify why a star can be regarded in dynamical and thermal equilibrium.

2.7.1. The virial theorem

The virial theorem gives a relation between the total gravitational energy and the total internal (thermal) energy. It is a very useful global diagnostic for a star.

Let us consider the total gravitational binding energy:

$$\begin{aligned}
 \Omega &= - \int_c^s \frac{GM_r}{r} dM_r \\
 &= - \int_c^s \frac{GM_r}{r} \rho 4\pi r^2 dr \\
 &= - \int_c^s \frac{GM_r}{r^2} \rho 4\pi r^3 dr \\
 &= \int_c^s \frac{dP}{dr} 4\pi r^3 dr \\
 &= \int_c^s 4\pi r^3 dP,
 \end{aligned} \tag{30}$$

where the integration limits refer to the appropriate central and surface values. The previous equation can be integrated by parts

$$\begin{aligned}
 \Omega &= P 4\pi r^3|_c^s - \int_c^s P d(4\pi r^3) \\
 &= -3 \int_c^s P dV,
 \end{aligned} \tag{31}$$

where the first vanishes because at the centre we have $r = 0$ while at the surface $P = 0$.

- Now for non-relativistic particles, the pressure is related to the internal energy density u by $P = 2u/3$, so we have

$$\Omega = -2 \int u dV \equiv -2U, \quad E_{\text{tot}} = \Omega + U = \frac{\Omega}{2} < 0 \tag{32}$$

where U and E_{tot} are the total internal energy and the total energy, respectively.

- For ultra-relativistic particles, $P = u/3$, and we have

$$\Omega = -3U, \quad E_{\text{tot}} = \Omega + U = 0 \quad (33)$$

Notice the total energy for ultra-relativistic particles is 0! So the system is unstable dynamically (if we add a small amount of energy to the system, it will escape to ∞). Very massive stars are dominated by radiation pressure (to be shown in §3), and so they are dynamically unstable!

2.7.2. Dynamical time

Let us consider a fun question: if we suddenly remove all the pressure of the gas, what will happen to the star? There is no longer any pressure to balance the gravity, the star must collapse. The question is how long it will take for the star to collapse to the center. This time-scale is often called the *free-fall* time-scale or *dynamical* time-scale.

Using eq. (10), but setting the pressure to zero, we find

$$\frac{d^2r}{dt^2} = -\frac{GM_r}{r^2}, \quad (34)$$

So we have

$$\frac{d^2r}{dt^2} = -\frac{GM_r}{r^2}, \quad (35)$$

Notice that for each shell, M_r is fixed.

The above equation can be solved exactly for each radius. However, we will satisfy ourselves by obtaining an approximate, order-of-magnitude estimate using dimensional analysis.

The left hand side has a dimension of $-r/t^2$, we use the negative sign because during the collapse the stellar radius decreases from r to zero. The right hand side has $-GM_r/r^2$. Dimensional analysis assumes that these two are approximately equal, so

$$-\frac{r}{t^2} \approx -\frac{GM_r}{r^2} \quad (36)$$

which gives a collapse time of

$$t = \left(\frac{r^3}{GM_r} \right)^{1/2} \quad (37)$$

Since $M = \bar{\rho}4\pi r^3/3$, where $\bar{\rho}$ is the average density, therefore we have

$$t = \left(\frac{3}{4\pi G\bar{\rho}} \right)^{1/2} \quad (38)$$

Exercise 2.2 Show that for a uniform sphere with density ρ , the free-fall time can be derived exactly, and is given by $t = (3\pi/32G\rho)^{1/2}$.

Our approximate solution is remarkably accurate, in fact, it is smaller than the exact result by only 10%.

For our sun, at the surface, we have $r = 7 \times 10^8$ m, $M = 2 \times 10^{30}$ kg. The free-fall time scale is

$$t = 27 \text{ minutes} \quad (39)$$

This is much shorter than the nuclear burning time scale of a solar-type star ($\sim 10^{10}$ years).

If there is any in-equilibrium in a star, the star can quickly adjust to an equilibrium state on the dynamical time-scale. As a result, the star can be viewed in most cases as in hydrostatic equilibrium.

Exercise 2.3 Estimate the dynamical time for a red giant with $r \approx 100R_{\odot}$, $M \approx 1M_{\odot}$. (answer: 18 days)

Exercise 2.4 Suppose the gravity of a star suddenly disappears. Obviously the gas will “explode” and flow away. Estimate the time-scale involved. (answer: $r/(P/\rho)^{1/2}$)

2.7.3. Kelvin-Helmholtz time-scale

In the early days of astrophysics, the energy source of stars was thought to be provided by the extraction of gravitational energy. The time scale that the gravitational energy can sustain the luminosity of a star is called the Kelvin-Helmholtz time-scale.

The total gravitational energy is given by (see §2.7.1)

$$\Omega = - \int_0^M \frac{GM_r}{r} dM_r \approx - \frac{1}{R/2} \int_0^M GM_r dM_r = - \frac{GM^2}{R} \quad (40)$$

where we have approximate r with $R/2$ in the second step.

From the Virial theorem (§2.7.1), we have

$$\Omega = -2U, \quad U = -\frac{\Omega}{2}. \quad (41)$$

Consider a proto-star contracts from a very large radius. It decreases steadily in gravitational energy. According to the virial theorem, exactly one half of the energy decrease will be used to increase the thermal energy. The other half will be lost by radiation. Thus the net energy reservoir available for radiation is equal to one half of the (absolute value of) gravitational energy.

The Kelvin-Helmholtz time-scale is then

$$t_{\text{KH}} = \frac{|\Omega|/2}{L} \approx \frac{GM^2}{2RL} \quad (42)$$

For the Sun, this is about 1.6×10^7 yr, which is much shorter than the current estimated age of the Sun, 4.5×10^9 yr. This has two important implications

- The Sun cannot be powered by the gravitational energy, in fact by a very large margin. We now know that the energy must be powered by nuclear burning.

- The Kelvin-Helmholtz time scale also turns out to approximately equal to the thermal-adjustment time-scale. In other words, the time it takes for a star to re-achieve thermal equilibrium if perturbed ². Since this time-scale is much shorter than the nuclear burning time scale, we can say that the star always adjusts itself such that the energy loss rate is balanced out by the energy generate rate in a star such as the Sun. i.e., the star is in thermal equilibrium.

Summary

In this section, we have derived the four basic stellar equations, and specified the boundary conditions. We highlighted that we need three key pieces of information, that is, equation of state, nuclear fusion reaction rate and coefficient of heat conductivity. We will discuss these in §3-5.

We also discussed several examples of using orders of magnitude estimates in astrophysics. It is an art that you should try to master.

²see Kippenhahn and Weigert , p.33