3. Equation of State

In the stellar interior, as we shall see, the temperature is so high that most elements are completely ionised. Therefore we have electrons, ions and photons at any radius $r$. They collide with each other very frequently and exchange energy, and thus local thermal equilibrium is achieved very quickly. So electrons, ions and photons have identical temperature locally. Notice that there is still a large-scale temperature gradient across the star, $T(r)$. However, a temperature gradient exists but we assume that it is small enough to be neglected for thermodynamic purposes. The local thermal equilibrium simplifies the problems tremendously.

The equation of state describes how the pressure changes with density and temperature

$$P = P(\rho, T, X_i)$$

We already know the equation of state for an ideal gas,

$$P = n k T,$$

where $n$ is the number density of particles, $k$ is the Boltzmann constant, and $T$ is the temperature.

The equation of state can depart from the classical ideal gas in three ways:

- The electrons can be highly relativistic.
- In some cases, the electrons are degenerate. In such cases, we have the so-called degenerate pressure to balance the gravity. The degenerate pressure is most important in very compact stars, such as white dwarfs and neutron stars.
- In massive stars, the pressure provided by radiation can be important.

Below we will establish a uniform framework to calculate the equation of state for all three cases.

3.1. General formalism

3.1.1. Density of states

Let us consider the density of quantum states in the phase space, $(\bar{x}, \bar{p})$. We know from Heisenberg uncertainty principle,

$$\delta x \delta p_x \geq \frac{\hbar}{2}$$

Hence we have

$$\delta x \delta p_x \delta y \delta p_y \delta z \delta p_z \geq \left(\frac{\hbar}{2}\right)^3$$

Physically, this means that a particle occupies at least a volume of of $(\hbar/2)^3$ in the phase space. So in a volume $V$ the number of states with momentum between $p$ and $p + dp$ is expected to be roughly

$$dN(p) \approx \frac{V \times 4\pi p^2 dp}{(\hbar/2)^3}$$
where $4\pi p^2 dp$ is the differential volume in the momentum space.

We would like to be more rigorous than the above rough estimate. Let us consider a box with side length $L_x, L_y, L_z$, then the quantum states in the box are described by standing waves. The wave function is given by

$$
\psi = A \sin \left( \frac{n_x \pi}{L_x} x \right) \sin \left( \frac{n_y \pi}{L_y} y \right) \sin \left( \frac{n_z \pi}{L_z} z \right) \equiv A \sin k_x x \sin k_y y \sin k_z z,
$$

where $A$ is a (fixed) normalisation constant. The wave number is given by

$$
k_x = \frac{n_x \pi}{L_x}, \quad k_y = \frac{n_y \pi}{L_y}, \quad k_z = \frac{n_z \pi}{L_z}
$$

(7)

The number of states with $k_x \to k_x + dk_x$, $k_y \to k_y + dk_y$, and $k_z \to k_z + dk_z$ is simply given by

$$
dN = dn_x dn_y dn_z = \frac{L_x}{\pi} dk_x \frac{L_y}{\pi} dk_y \frac{L_z}{\pi} dk_z = \frac{V}{\pi^3} dk_x dk_y dk_z,
$$

(8)

where the volume $V$ is given by $L_x L_y L_z$.

The number of quantum states with wave number magnitude between $k$ to $k + dk$ is then

$$
dN(k) = \frac{V}{\pi^3} \frac{4\pi k^2}{8} dk,
$$

(9)

the factor of 8 arises because we are only counting the states in the quadrangle with positive $n_x, n_y$ and $n_z$.

Now the wave number is related to the momentum by

$$
p = \hbar k
$$

(10)

Hence the density of states between $p$ to $p + dp$ is

$$
dN(p) = \frac{V}{\pi^3} \frac{4\pi p^2}{8} dp \times \hbar^3 = \frac{V}{\hbar^3} 4\pi p^2 dp
$$

(11)

Notice that this is very similar to our guess (eq. 5), other than a constant factor.

The density of states between $p$ to $p + dp$ per unit volume is then

$$
g(p)dp = \frac{dN(p)}{V} = \frac{4\pi p^2}{\hbar^3} dp
$$

(12)

So far we have neglected one important fact: for each state with given momentum, there could be several quantum states with different spins or polarisations. If the number of spin states is $g_s$, then the modified density of states per unit volume is

$$
g(p)dp = g_s \frac{4\pi p^2}{\hbar^3} dp
$$

(13)

$g_s$ is also often called statistical weight.

- For electrons, protons, and neutrons, they have spin $s = 1/2$, and $g_s = 2s + 1 = 2$.
- For neutrinos, its spin is $1/2$, but we have $g_s = 1$, since we have only left-handed neutrinos due to parity violation.
- For photons, the spin is 1, but it has only two polarisations (perpendicular to the direction of propagation), so $g_s = 2$. 
3.1.2. Occupation numbers

From statistical mechanics, we know that the occupation number for each state is given by

\[ f(E) = \frac{1}{e^{(E-\mu)/kT} \pm 1} \]  

(14)

where \( E \) is the energy, \( k \) is the Boltzmann constant, \( T \) is the temperature, and \( \mu \) is the chemical potential. Note that

- the energy is in general a function of momentum, \( E(p) \).
- the plus sign is for Fermions, such as electrons and protons. Fermions have spins 1/2, 3/2 etc., they obey the Fermi-Dirac statistics.
- the minus sign is for Bosons, such as photons. They have integer spins, and obey the Bose-Einstein statistics.

The number density of particles with momentum between \( p \) to \( p + dp \) can be obtained by multiplying the density of states and the occupation number, which gives

\[ n(p)dp = g_s \frac{4\pi p^2 dp}{\hbar^3} \frac{1}{e^{(E-\mu)/kT} \pm 1} \]  

(15)

The total number density of particles can be obtained by integrating over all the momentum

\[ n = \int_0^\infty n(p)dp. \]  

(16)

3.1.3. Internal energy, velocity and kinetic energy density

As our particles can be relativistic, so we have to use a general formula for the relation between \( E \) and \( p \). This is provided by special relativity, for free particles

\[ E = \sqrt{m^2 c^4 + c^2 p^2}, \]  

(17)

where \( m \) is the rest-mass of the particle. The total energy has of course two parts, the rest mass energy, and the kinetic energy

\[ E = mc^2 + E_k \]  

(18)

There are two simple asymptotic limits:

- when the particle is non-relativistic (NR), in this case,

\[ E = mc^2 \sqrt{1 + \frac{p^2}{m^2 c^2}} \approx mc^2 \left(1 + \frac{p^2}{2m^2 c^2}\right) \approx mc^2 + \frac{p^2}{2m} \]  

(19)

This means the kinetic energy is

\[ E_k = \frac{p^2}{2m} \]  

(20)

in agreement with Newtonian mechanics.
• when the particle is ultra-relativistic (UR), i.e., \( p \gg mc \). In this case, the first term in eq. (17) can be neglected and we have

\[
E = cp, \quad E_k = cp
\]  

(21)

Notice this is identical to the relation for photons, as it should be.

From special relativity, we have \( p = \gamma mv \) and \( E = \gamma mc^2 \) (\( \gamma \) is the Lorentz factor), and so

\[
v = \frac{c^2 p}{E}
\]  

(22)

Substituting eq. (17) into the previous equation, we obtain

\[
v = \frac{cp}{(m^2c^2 + p^2)^{1/2}}
\]  

(23)

It is easy to show that again we have two asymptotic limits:

• for the non-relativistic (NR) case, \( v \approx p/m \)

• for the ultra-relativistic (UR) case, \( v \approx c \).

The kinetic energy per unit volume, also called kinetic energy density, can be obtained by integrating over the product of the number density of particles and their kinetic energy:

\[
u = \int_0^\infty E_k(p)n(p)dp.
\]  

(24)

3.1.4. Pressure

To obtain the equation of state, we need to have an expression of the pressure. This can be derived as follows.

Suppose we have particles inside a box. Let us consider the wall with its normal along the positive \( x \)-direction. The wall has a surface area of \( A \). Now pressure is simply the force divided by area

\[
P = \frac{F}{A}, \quad F = \frac{dp}{dt}
\]  

(25)

Let us consider the simplest case, where the particles all move in the +\( x \) direction, they all have momentum \( p \), and the number density of particles is \( n \). Over a period of \( dt \), the particles within volume \( Av(p)dt \) will collide with the surface. The total number of particles that will collide is

\[
dN = nAv(p)dt.
\]  

(26)

These particles will be bounced back with the same momentum but in the opposite direction, so the momentum change is \( 2p \) for each collision. Therefore the total momentum change will be

\[
dp = (2p)dN = (2p) \times nAv(p)dt,
\]  

(27)

and hence the force is

\[
F = \frac{dp}{dt} = 2 \ n \ p \ A \ v(p)
\]  

(28)
Therefore from eq. (25), the pressure is given by
\[ P = F \frac{A}{A} = 2 \ n \ p \ v(p) \] (29)

Now for an isotropic particle distribution, there will be on average \( \frac{1}{6} \) of particles moving in the positive \( x \) direction, and hence
\[ P = \frac{1}{3} \ n \ p \ v(p), \] (30)

Also the particles may have different momenta, and therefore we need to integrate over the momentum distribution.
\[ P = \int_{0}^{\infty} \frac{1}{3} \ p \ v(p) \ n(p) \ dp. \] (31)

This is our final expression for the pressure, valid at all velocities and for all momentum distributions, as long as the particle distribution is isotropic.

**Exercise 3.1** Show rigorously by integrating over the solid angles that the above expression for pressure is correct.

We can immediately derive a relation between the pressure and kinetic energy density.

- For the NR case, \( v = \frac{p}{m} \), so
\[ P = \int_{0}^{\infty} \frac{1}{3} \ p \ \left( \frac{p}{m} \right) \ n(p) \ dp = \int_{0}^{\infty} \frac{1}{3} (2E_k) \ n(p) \ dp = \frac{2}{3} u \] (32)

  in other words, \( u = 3P/2 \).

- for the UR case, \( v = c \), so
\[ P = \int_{0}^{\infty} \frac{1}{3} \ p \ c \ n(p) \ dp = \int_{0}^{\infty} \frac{1}{3} (E_k) \ n(p) \ dp = \frac{1}{3} u \] (33)

  in other words, \( u = 3P \), for all ultra-relativistic particles, such as photons.

*Note these two relations are very general, independent of the momentum distribution!*

Eqs. (16) and (31) in principle allow us to calculate the pressure as a function of the number density for any gas made of identical particles in local thermal equilibrium.

Notice that we have so far neglected the interaction energy between particles, i.e., we assume that there are interactions between particles to establish thermal equilibrium, but they are weak enough to be neglected. This assumption is normally satisfied inside stars. So we will adopt this assumption throughout this course.

**Exercise 3.2** Show that for the Sun, the interaction energy is indeed negligible compared with thermal energy. (Hint: estimate the Coulomb energy and thermal energy [65eV and 1.3keV]).

In the following, we shall apply the above general formalism to derive the equation of state for classical gas, degenerate gas, and radiation.
3.2. Maxwell distribution

This is the case when

\[ mc^2 \gg E - \mu \gg kT, \]  

(34)
i.e., the particles are non-relativistic, and the exponential term in the denominator of eq. (14) is very large. This means the occupation number is much less than 1 in each quantum state. The term \( \pm 1 \) can be neglected, and hence the classical distribution for Fermions and Bosons will be identical.

With the approximation in eq. (34), we have, for the classical gas distribution,

\[ v(p) \approx \frac{p}{m}, \quad E = mc^2 + E_k \approx mc^2 + \frac{p^2}{2m} \]  

(35)

and eq. (15) simplifies to

\[ n(p)dp = g_s \frac{4\pi p^2}{h^3} dp \ e^{-(mc^2+\frac{p^2}{2m}-\mu)/kT} = Ap^2 dp \ e^{-p^2/2mkT}, \]  

(36)

where we have collected all the constants into \( A \)

\[ A = g_s \frac{4\pi}{h^3} e^{-(mc^2-\mu)/kT}. \]  

(37)

The number density of particles is

\[ n = \int_0^\infty n(p)dp = \int_0^\infty Ap^2 dp \ e^{-p^2/2mkT} = A(2mkT)^{3/2} \int_0^\infty dx \ x^2 \ e^{-x^2} \]  

(38)

where we substituted \( x^2 = p^2/2mkT \). Now the last integral is equal to \( \pi^{1/2}/4 \), so we have

\[ n = A(2mkT)^{3/2} \frac{\pi^{1/2}}{4} = g_s \left( \frac{2\pi mkT}{h^2} \right)^{3/2} e^{-(mc^2-\mu)/kT} \]  

(39)

This equation also tells us how to derive the chemical potential once we know the temperature and number density of gas.

Similarly, we can derive the expression for the pressure, using eq. (31),

\[ P = \frac{1}{3} \int_0^\infty Ap^2 \frac{p^2}{m} dp \ e^{-p^2/2mkT} = \frac{A}{3m} \int_0^\infty p^4 dp \ e^{-p^2/2mkT} = \frac{A}{3m} (2mkT)^{5/2} \int_0^\infty dx \ x^4 \ e^{-x^2} \]  

(40)

The last integral is equal to \( 3\pi^{1/2}/8 \), and hence

\[ P = A \pi^{1/2} \frac{(2mkT)^{5/2}}{8m} \]  

(41)

Dividing eq. (41) by eq. (39), we obtain

\[ P/n = kT. \]  

(42)
i.e., \( P = nkT \). This is the familiar equation of state for an ideal gas.

It is interesting to check what the approximation (eq. 34) means physically. We will show below, that the approximation requires the separation between particles to be much larger than the typical de Broglie wavelength of a particle. This can be seen as follows.
From eq. (39), we have

\[ n = g_s \left( \frac{2\pi m kT}{\hbar^2} \right)^{3/2} e^{-\frac{(mc^2-\mu)}{kT}}. \]  

(43)

In our approximation, \( E - \mu \approx mc^2 - \mu \gg kT \), the last exponential term is therefore much smaller than one, which implies that

\[ n \ll \left( \frac{2\pi m kT}{\hbar^2} \right)^{3/2} \equiv n_Q, \]

(44)

where \( n_Q \) is called the quantum concentration or critical number density and we have ignored \( g_s \) which is of the order of unity. Notice that the term in the bracket, \( 2\pi m kT/\hbar^2 \) in eq. (44) is related to the de Broglie wavelength. This is so because a particle has a typical kinetic energy \( p^2/2m \sim kT \), i.e., \( p \approx (2mkT)^{1/2} \), and therefore the de Broglie wavelength is

\[ \lambda = \frac{\hbar}{p} \approx \frac{\hbar}{(2mkT)^{1/2}}. \]

(45)

The approximation in Eq. (44) can then be rewritten as

\[ n \ll \left( \frac{\pi}{\lambda^2} \right)^{3/2} = \frac{\pi^{3/2}}{\lambda^3}, \]

(46)

Since the average separation between particle, \( d \), is \( d \sim n^{-1/3} \), the above equation means

\[ d \gg \lambda, \]

(47)

i.e., the separation between particles must be much larger than the de Broglie wavelength. In quantum mechanics, we have particle-wave duality, and particles should be viewed as a sort of fuzzy ball with size \( \sim \lambda \). When the separation is much larger than this size, the wave nature of particles can be completely ignored. In other words, the particles behave just like classical (point) particles, and hence they follow the classical Maxwell distribution.

There are two further interesting points

1. For photons, they never follow the Maxwell distribution, because as we will see, their chemical potential is zero. There are always photons that violate the classical condition.

2. The de Broglie wavelength is larger for light particles, so the critical density for quantum behaviour is lower for light particles. When a star evolves, the density at the centre usually becomes higher, and the lightest particles (electrons) are the first to reach the critical density, and therefore they are the first to exhibit the quantum behaviour. In fact, ions can always be approximated as an ideal gas in most regimes of interests.

### 3.2.1. Mean molecular weights

So far, we have only discussed the number density of particles (\( n \)), we still need to relate \( n \) to the density \( \rho \). This relation depends on the chemical composition of stars. It is conventional to write
• $X$: as the fraction of mass in hydrogen
• $Y$: as the fraction of mass in helium
• $Z$: as the fraction of mass in elements heavier than hydrogen and helium. These heavier elements are normally (misleadingly) called “metals” in astrophysics, although most of the elements are in fact, C, N, and O. $Z$ is usually referred to as metallicity.

Of course, we have

$$X + Y + Z = 1 \quad (48)$$

For a typical main sequence star, $X = 0.7$, $Y = 0.28$, and $Z = 0.02$. Hence, most mass is in hydrogen and helium, while a very small fraction is in heavier elements.

In most stellar applications, it is safe to assume that the stars are fully ionised, the only exceptions are stellar atmospheres. So the pressure is provided by not only electrons but also ions. To use the ideal gas equation of state, we must count the number of particles in the gas.

• Each hydrogen atom has a mass of about $m_p$.
• Each helium atom has a weight of about $4m_p$.
• Each heavier element with a nuclear charge $Z^*$ has $Z^*$ protons and approximately equal number of neutrons. Therefore each heavier element contributes a mass of about $2Z^* m_p$.

The average charge of all heavier elements is approximately $\langle Z^* \rangle \approx 8$.

Let us consider the number of atoms in a differential volume $dV$. The total mass will be $\rho dV$, and hence the mass of hydrogen, helium and heavier elements are $X \rho dV$, $Y \rho dV$, and $Z \rho dV$. In this amount of matter, there are $X \rho dV / m_p$ hydrogen nuclei, $Y \rho dV / 4 m_p$ of helium, and approximately $Z \rho dV / (2 \langle Z^* \rangle)$ heavier element nuclei. Therefore the total number of ions is

$$dN_i = \frac{\rho dV}{m_p} \left( X + \frac{Y}{4} + \frac{Z}{2 \langle Z^* \rangle} \right) \quad (49)$$

And hence the number density of ions is

$$n_i = \frac{dN_i}{dV} = \frac{\rho}{m_p} \left( X + \frac{Y}{4} + \frac{Z}{2 \langle Z^* \rangle} \right) \approx \frac{\rho}{m_p} \left( X + \frac{Y}{4} + \frac{Z}{16} \right) \quad (50)$$

Similarly, we can estimate the number density of electrons. A fully ionised hydrogen atom provides one electron per nucleon. A fully ionised helium atom provides one electron per two nucleons. A heavy element with charge $Z^*$ provides $Z^*$ electrons per nucleus. In a typical case, it has $A \approx 2Z^*$ nucleons, i.e., we get one electron per 2 nucleons, just like a helium atom. The number density of electrons is therefore

$$n_e \approx \frac{\rho}{m_p} \left( X + 2 \times \frac{Y}{4} + \frac{Z}{2 \langle Z^* \rangle} \right) = \frac{\rho}{m_p} \frac{1 + X}{2} . \quad (51)$$

So the total number density of particles is

$$n = n_e + n_i = \frac{\rho}{m_p} (2X + 0.75Y + 0.56Z) \quad (52)$$
It is conventional to define the mean molecular weight $\mu$ per particle, mean molecular weight per electron, and mean molecular per ion, as,

$$\mu \equiv \frac{\rho}{n m_p} \approx \frac{1}{2X + 0.75Y + 0.56Z}$$  \hfill (53)

$$\mu_e \equiv \frac{\rho}{n_e m_p} \approx \frac{2}{1 + X}$$  \hfill (54)

$$\mu_i \equiv \frac{\rho}{n_i m_p} \approx \frac{1}{X + Y/4 + Z/16}$$  \hfill (55)

One should not confuse the mean molecular weight with the chemical potential, which is denoted by the same symbol. The meaning of $\mu$ should be clear from the context.

With the newly defined mean molecular weights, we may write the equation of state as

$$P_e = n_e kT = \frac{\rho}{\mu_e m_p} kT$$  \hfill (56)

$$P_i = n_i kT = \frac{\rho}{\mu_i m_p} kT$$  \hfill (57)

And the total pressure as

$$P = P_e + P_i = (n_e + n_i) kT = \frac{\rho}{\mu m_p} kT$$  \hfill (58)

The corresponding kinetic energy densities are given by

$$u_e = 1.5 P_e, \quad u_i = 1.5 P_i, \quad u = 1.5 P.$$  \hfill (59)

**Example 3.1** Let us calculate the mean molecular weights per particle, per electron, and per nucleon for the sun.

For the Sun, $X = 0.7, Y = 0.28, Z = 0.02$, and we have

$$\mu = 0.62, \quad \mu_e = 1.18, \quad \mu_i = 1.3$$  \hfill (60)

**Example 3.2** Let us check whether the gas at the centre of the Sun is described by ideal gas law,

$$\rho_c = 1.5 \times 10^5 \text{ kg m}^{-3}, \quad T_c = 1.6 \times 10^7 \text{ K.}$$  \hfill (61)

We have, $\mu_e = 2/(1 + X) \approx 1.2$, therefore, the number density of electrons is

$$n_e = \frac{\rho_c}{(\mu_e m_p)} = 7.6 \times 10^{31} \text{ m}^{-3}$$  \hfill (62)

The average separation between the electrons is about

$$d \approx n_e^{-1/3} = 2.4 \times 10^{-11} \text{ m} < \text{Bohr radius}$$  \hfill (63)

The critical number density is

$$n_Q = \left(\frac{2\pi m_e kT_e}{h^2}\right)^{3/2} = 1.4 \times 10^{32} \text{ m}^{-3}$$  \hfill (64)

Hence the actual number density of electrons at the centre of Sun is just below the critical value, so it can be approximated as an ideal gas. However, a rigorous treatment requires us to account for the small but significant deviation from the classical behaviour.
Example 3.3 Early evolution of the Sun

When a star just initiates the hydrogen nuclear burning, it is usually called a zero-age main sequence (ZAMS) star. For example, the Sun started out as a ZAMS star. Let us consider how the central density and temperature evolves as hydrogen is burned in the core of the Sun. As hydrogen is converted into helium, the hydrogen mass fraction $X$ decreases while $Y$ increases, from eq. (53), $\mu$ increases, so the central pressure decreases if the central temperature and density are held as constant. So in order for the pressure to balance the gravity, the star must contract to increase its density and temperature. As the temperature increases, the energy generation rate is also increased, and more energy is transported to the surface as well. So in the HR diagram, the Sun evolves from the ZAMS position to the left (increasing temperature) and up (increasing luminosity). Here, we have a paradox, as the fuel (hydrogen) is gradually consumed, the luminosity of the Sun actually increases!

3.3. Fermi-Dirac distribution

We have already discussed that stars radiate, and therefore they lose energy. As a result of this, the core of stars usually contract, so the central density increases over time. When the density becomes high enough, the classical behaviour for electrons first breaks down. To see this, we need to examine how the central density and the critical density scales with radius as the star contracts.

The central density, from dimensional analysis, should scale as

$$n \propto \frac{M}{R^3} \propto R^{-3},$$

(65)

where $M$ and $R$ are the total mass and stellar radius of the star.

To estimate the critical density, we need an estimate of the central temperature. We know from hydrostatic equilibrium,

$$\frac{dP}{dr} = -\frac{GM(r)}{r^2} - \rho(r)$$

(66)

We adopt the approximation that $\rho \approx \rho_c \approx \langle \rho \rangle \approx M/R^3$, then from dimensional analysis, we have

$$\frac{P_c}{R} \approx \frac{GM}{R^2} \cdot \rho \approx \frac{GM M}{R^2 R^3} = \frac{GM^2}{R^5} \rightarrow P_c = \frac{GM^2}{R^4}$$

(67)

Now if the central gas observes the ideal gas law, then

$$P_c = \rho_c \frac{kT_c}{\mu m_p} = M \frac{kT_c}{R^3 \mu m_p}$$

(68)

The previous equation then leads to

$$T_c \approx \frac{GM \mu m_p}{kR} \propto R^{-1}$$

(69)

Hence the critical number density for electrons is given by

$$n_Q = \left(\frac{2\pi m_e kT_c}{h^2}\right)^{3/2} \propto T_c^{3/2} \propto R^{-3/2}$$

(70)
From eq. (65), the physical central density as $R^{-3}$. So as the star contracts, the central density rises faster than the critical density required for quantum behaviour. Notice also that the critical number density for proton degeneracy is higher by a factor of $(m_p/m_e)^{3/2}$. As a result, the electrons will first become degenerate. In addition, if the density is high enough, the electrons can become relativistic as well.

### 3.3.1. $T = 0$ degenerate electron gas

In this section, we will first treat the case when the $n \gg n_Q$. This condition also means the temperature satisfies

$$T \ll \frac{\hbar^2}{2\pi m k} n_e^{2/3} \equiv T_{cr} \approx 10^7 \text{K} \left(\frac{n_e}{10^{32} \text{m}^{-3}}\right)^{2/3}$$

(71)

If the temperature is much lower than the critical temperature, $T_{cr}$, then the gas can be regarded as “cold”. In this subsection, we will idealise this cold gas as having $T = 0$; finite temperature degenerate gas is more difficult to treat and therefore will be only briefly discussed in the next subsection.

Such a cold gas of electrons is often called degenerate because electrons have fallen into the lowest quantum energy states. Electrons are fermions – they obey Pauli exclusion principle, and so the electrons must be distributed such that the energy levels up to a certain level is completely filled. The highest energy level is called Fermi-energy, $E_F$, and the corresponding the momentum, is called Fermi momentum, $p_F$. From special relativity, we have

$$E_F = \sqrt{c^2 p_F^2 + m_e^2 c^4}$$

(72)

where $m_e$ is the rest mass of electron.

The occupation number is then

$$f(E) = \begin{cases} 1 & \text{if } E \leq E_F \\ 0 & \text{if } E > E_F \end{cases}$$

or equivalently

$$f(p) = \begin{cases} 1 & \text{if } p \leq p_F \\ 0 & \text{if } p > p_F \end{cases}$$

For electrons, $g_s = 2$, so we have for the number density of electrons

$$n_e = \int_0^\infty n(p)dp = \int_0^{p_F} \frac{8\pi p^2 dp}{h^3} = \frac{8\pi p_F^3}{3h^3}.$$ 

(73)

This provides a direct link between the number density of electrons and the Fermi momentum. The density of gas is related to $n_e$ via eq. (54)

$$\rho = n_e \mu_e m_p = \frac{8\pi}{3} \left(\frac{p_F}{h}\right)^3 \mu_e m_p$$

(74)

where $\mu_e = 2/(1 + X)$ is the mean molecular weight per electron for a fully ionised gas and $X$ is again the fraction of mass in hydrogen. The previous equation can also be rewritten as

$$\rho = \rho_0 \mu_e x_F^3,$$

(75)
where \( \rho_0 \) is a constant and has a dimension of density, given by
\[
\rho_0 = \frac{8 \pi}{3} \left( \frac{m_e c}{h} \right)^3 m_p = 0.98 \times 10^9 \text{ kg m}^{-3}
\]  
and \( x_F = p_F/m_e c \).

For the pressure, from eq. (31), and taking into account the occupation number, we have
\[
P_e = \int_0^{p_F} \frac{1}{3} p v(p) n(p) \, dp = \int_0^{p_F} \frac{1}{3} p \frac{p}{m_e \sqrt{1 + p^2/(m_e^2 c^2)}} \frac{8 \pi p^2 \, dp}{h^3}
\]
The above equation can be rewritten as
\[
P_e = \frac{8 \pi}{3 h^3 m_e (m_e c)^5} \int_0^{x_F} \frac{x^4 \, dx}{\sqrt{1 + x^2}} \equiv B \int_0^{x_F} \frac{x^4 \, dx}{\sqrt{1 + x^2}}
\]
where we have substituted \( x = p/(m_e c) \) and \( x_F = p_F/(m_e c) \),
\[
B = \frac{8 \pi}{3 h^3 m_e (m_e c)^5} = \frac{8 \pi}{3} \left( \frac{m_e c}{h} \right)^3 m_e c^2 = 4.80 \times 10^{22} \text{ J m}^{-3}
\]
Similarly, for the kinetic energy density, we have
\[
u_e = \int_0^{\infty} E_k(p) n(p) \, dp = \int_0^{p_F} \left[ \sqrt{c^2 p^2 + m_e^2 c^4} - m_e c^2 \right] \frac{8 \pi p^2 \, dp}{h^3}
\]
The previous equation can be rewritten as
\[
u_e = 3 u_0 \int_0^{x_F} \left[ (1 + x^2)^{1/2} - 1 \right] x^2 \, dx = 3 u_0 \int_0^{x_F} \frac{x^4 \, dx}{(1 + x^2)^{1/2} + 1}
\]

Eqs. (75), (78), and (81) complete the equations of state for a completely degenerate gas in parametric form. These equations can be integrated analytically, but we will be satisfied by considering two limiting cases: when \( x_F \ll 1 \), i.e., when the gas is degenerate and non-relativistic; when \( x_F \gg 1 \), i.e., when the gas is degenerate and ultra-relativistic.

For \( x_F \ll 1 \), we have \( 1 + x^2 \approx 1 \), so we obtain
\[
\rho = \rho_0 \mu_e x_F^3, \quad P_{e, \text{NR}} = \frac{1}{5} u_0 x_F^5 = K_{\text{NR}} (\rho/\mu_e)^{5/3}, \quad \nu_{e, \text{NR}} = \frac{3}{10} u_0 x_F^5 = \frac{3}{2} P_{e, \text{NR}}
\]
where \( K_{\text{NR}} = 0.99 \times 10^7 \text{ J m/kg}^{5/3} \) and the subscript ‘NR’ stands for non-relativistic. The relation between the pressure and kinetic energy density is just that expected for any non-relativistic gas.

For \( x_F \gg 1 \), we have \( 1 + x^2 \approx x^2 \), so we obtain
\[
\rho = \rho_0 \mu_e x_F^3, \quad P_{e, \text{UR}} = \frac{1}{4} u_0 x_F^4 = K_{\text{UR}} (\rho/\mu_e)^{4/3}, \quad \nu_{e, \text{UR}} = \frac{3}{4} u_0 x_F^5 = 3 P_{e, \text{UR}}
\]
where \( K_{\text{UR}} = 1.23 \times 10^{10} \text{ J m/kg}^{4/3} \) and the subscript ‘UR’ stands for non-relativistic. The relation between the pressure and kinetic energy density is just that expected for any non-relativistic gas.

The transition between degenerate non-relativistic gas and degenerate ultra-relativistic gas should be when the Fermi momentum is about equal to \( m_e c \), i.e., when \( x_F = 1 \), this implies a density of, from eq. (75),
\[
\rho = \rho_0 \mu_e = 0.98 \times 10^9 \mu_e \text{ kg m}^{-3}
\]
In the transition region, we can either use the exact parametric formula or use the following approximation

\[ P_{e,d} \approx (P_{e,NR}^2 + P_{e,UR}^2)^{-1/2}, \]  

This interpolation formula chooses the smaller of the two limiting case formulae. It is accurate to about 2%.

**Exercise 3.3** For \( x_F = 1 \), using eq. (85) to find the approximate pressure. Compared with the exact value \( 1.6015 \times 10^{23} \text{Pa} \), what is the error in your approximation? (answer:1.66%)

### 3.3.2. Finite temperature electron gas

We already know two equations of state for finite temperature electron for non-relativistic electron gas.

The first case is when the gas is non-degenerate and non-relativistic. This occurs when the gas density is low and the temperature is high. More precisely, when \( m_e c^2 \gg E - \mu \gg kT \), the electrons behave as a classical gas, and they follows Maxwell distribution, and hence

\[ P_{e,nd} = \frac{\rho}{\mu_e m_p} kT, \]  

where the subscript refers to non-degenerate electron gas.

The second case is when the electrons are degenerate and when the gas can be regarded as “cold” (\( T \ll T_{cr} \)). In this case, the equation of state can be well approximated by our \( T = 0 \) completely degenerate gas, as given in eq. (85).

The difficulty arises when we have a non-relativistic, and partially degenerate regime, i.e., when \( E - \mu \approx kT \), there is no analytical expression, but one can use the following approximate formula that is accurate to about 2%:

\[ P_e \approx \left[ P_{e,nd}^2 + P_{e,d}^2 \right]^{1/2}, \]  

where \( P_{e,d} \) is given by eq. (85). The last formula selects the larger of the two limiting case formulae.

Let us now find the region in the density-temperature plane where there is a transition from non-degenerate to degenerate electron non-relativistic gas. This may be estimated by setting \( P_{e,nd} = P_{e,d} \approx P_{e,NR} \), which gives

\[ \frac{\rho}{\mu_e m_p} kT \approx .99 \times 10^7 \left( \frac{\rho}{\mu_e} \right)^{5/3} \rightarrow T \approx 1.2 \times 10^3 \mu_e \left( \frac{\rho}{\mu_e} \right)^{2/3} \text{K}. \]  

Fig. 1 summarises various regimes of electron pressure in the temperature and density plane.

#### 3.3.3. Electrons in the Sun and helium flash

We have already shown in §3.2 that at the centre of Sun, the gas can be approximated as an ideal gas. Now, as the Sun evolves, the central hydrogen will fuse into helium, and eventually
(a central helium core will form where there is no longer hydrogen available for nuclear burning. Afterwards, the heat that leaks out as radiation cannot be replenished by hydrogen fusion reactions. So the core contracts as it loses energy. The temperature rises due to the work done by gravity compression. The temperature of the hydrogen shell surrounding the helium core will also rise, eventually the temperature of the hydrogen shell is high enough to burn hydrogen. The energy generated in the shell is much higher than that generated by core hydrogen burning along the main sequence. Not all the large nuclear energy can be radiated. In fact, part of the energy is used to push the envelope outer-ward, as a result, the star will become hundreds of times larger than the present-day Sun. In other words, the Sun will then enter the red-giant phase in the HR diagram.

Through this phase, eventually the core of the Sun will reach a temperature of $T \approx 10^8 K$ and a number density of electrons of $n_e \approx 3 \times 10^{34} m^{-3}$. Let us see whether the electrons are degenerate in this case. The critical number density is

$$n_Q = \left( \frac{2\pi m_e kT}{h^2} \right)^{3/2} \approx 2 \times 10^{33} m^{-3}. \quad \text{(89)}$$

i.e., the actual physical number density is higher than the critical number density, therefore the electrons are highly degenerate in the helium core.

Although the temperature of the electrons is low by the requirement of degeneracy, it is high enough to eventually initiate the helium burning in the core, $3\alpha \rightarrow ^{12}C$. And this triggers a very violent event, the so-called helium flash.
In normal materials, when the temperature increases, the pressure increases, it expands and pushes the material of the surrounding gas outward, so the temperature drops as it loses energy. Such systems are thermally stable.

However, the helium burning occurs in a degenerate core, the system is thermally unstable. This can be seen as follows, the energy generation rate is extremely sensitive to the temperature, in fact,

$$\epsilon \propto \rho_{He}^2 T^{40}$$

For example, if the temperature rises by 10%, the energy generate rate increases by a factor of 45! When the helium burning reaction starts, the energy liberated increases the temperature locally, but because of the degeneracy, the rising temperature does not increase the pressure, the latter means the core cannot expand and cool itself. The increasing temperature means the reaction $3\alpha \rightarrow ^{12}\text{C}$ runs faster, and so on, so we have a thermal runaway. The luminosity can increase by a huge factor in a few minutes. Eventually the core degeneracy is lifted by expansion and the star becomes thermally stable again. Because this flash happens so fast, it is not easy to follow the evolution even with a large modern computer. In fact the details of how the degeneracy is lifted are still not completely understood.

The Sun does not have sufficient mass to go beyond helium burning. So it will end with a carbon-oxygen (CO) white dwarf, where the electron degeneracy is very important. We will discuss the structure of white dwarfs in much greater detail later.

3.3.4. Electrons in massive stars

Let us consider the importance of degeneracy as a function of the stellar mass. We have already shown that

$$T_c \approx \frac{GM\mu m_p}{R} \propto M^{2/3} \rho^{1/3},$$

where we have used $R \propto M^{1/3} \rho^{-1/3}$.

Since the central temperature of nuclear burning is fairly insensitive to the mass, this implies that the central density at given $T_c$ is lower for more massive stars. The degeneracy in massive stars is therefore less important. In fact, for stars with $M > 10M_\odot$, the degeneracy of electrons is never important in most stages of stellar evolution. For stars with $M \gtrsim 2M_\odot$, the helium ignition is relatively “peaceful”, i.e., no helium flash occurs.

However, at their end stage of evolution, they explode as supernovae. And for this stage, the degeneracy of electrons again plays a crucial role, and we will return to this at the later part this course.

3.4. Planck distribution

Photons have zero rest-mass, so they can be created and destroyed, and hence the number of photons is not conserved. A direct consequence of this is that the chemical potential $\mu$ for photons is zero.
This can be seen as follows. From thermodynamics, we have

$$dE = TdS - PdV + \mu dN,$$

where $S$ and $N$ are the entropy and the number of particles. For fixed energy and volume, the photons are created such that the entropy is maximum. This implies

$$\frac{\partial S}{\partial N}_{E,V} = -\frac{\mu}{T} = 0 \rightarrow \mu = 0$$

i.e., a photon gas in thermal equilibrium has zero chemical potential.

For photons, we have $g_s = 2$, $v = c$, $E = cp$, the number density of photons from $p$ to $p + dp$ is

$$n(p)dp = \frac{8\pi p^2 dp}{h^3} \frac{1}{e^{cp/kT} - 1}.$$ (94)

For photons, $p = h\nu/c$, so the number density of photons between $\nu$ and $\nu + d\nu$ is

$$n(\nu)d\nu = n(p)dp = \frac{8\pi\nu^2 d\nu}{c^3} \frac{1}{e^{h\nu/kT} - 1},$$ (95)

The energy density per unit frequency is then

$$U_\nu = n(\nu)h\nu = \frac{8\pi h\nu^3 d\nu}{c^3} \frac{1}{e^{h\nu/kT} - 1}.$$ (96)

The total radiation energy density is

$$u_r = \int_0^\infty U_\nu d\nu = \frac{8\pi h}{c^3} \left(\frac{kT}{h}\right)^4 \int_0^\infty \frac{x^3}{e^x - 1}dx$$

where we have used $x = h\nu/kT$. The last integral is equal to $\pi^4/15$, and therefore we have

$$u_r = \frac{8\pi h}{c^3} \left(\frac{kT}{h}\right)^4 \frac{\pi^4}{15} \equiv aT^4$$ (98)

where $a = 8\pi^5k^4/(hc)^3 = 7.565 \times 10^{-16}$ JK$^{-4}$ m$^{-3}$.

The pressure exerted by photons is called the radiation pressure, and is given by

$$P_t = \frac{1}{3}u_r = \frac{1}{3}aT^4,$$ (99)

because photons are ultra-relativistic.

**Exercise 3.4** In §3.1.4, the general expression was derived under the assumption that when a particle collides with a wall, the particle bounces back. Is this really what happens when a photon collides with the wall in a black box? If not, why can we still apply the formulae?
move in the right direction; the factor $\cos \theta$ comes from the reduction of effective surface area due to inclination. The fraction of isotropic photons within $\theta \to \theta + d\theta$ and $\phi \to \phi + d\phi$ that will eventually pass through the surface is

$$\frac{d\Omega}{4\pi} = \frac{\sin \theta \, d\theta \, d\phi}{4\pi}.$$  \hfill (100)

So the energy passing through the surface is

$$dE = \int u_r (dA \, c \, dt \, \cos \theta) \frac{d\Omega}{4\pi}.$$  \hfill (101)

If the radiation is isotropic, then this integral will be zero if we integrate over all directions, due to symmetry. We are, however, more interested in calculating flux contributed by one hemisphere of photons, i.e., we restrict $\theta$ from 0 to $\pi/2$. We have

$$f \equiv \frac{dE}{dt \, dA} = \int_0^{2\pi} d\phi \int_0^{\pi/2} c \, u_r \cos \theta \, \sin \theta \frac{d\theta}{4\pi} = \frac{1}{4} c \, u_r$$  \hfill (102)

An observer just outside the blackbody sphere will only see the photons from one hemisphere, and the flux is just

$$f = \frac{c u_r}{4} = \frac{c}{4} a T^4 = \sigma T^4,$$  \hfill (103)

where $\sigma = ac/4 = 5.67 \times 10^{-8} \text{W K}^{-4} \text{m}^{-2}$ is the Stefan-Boltzmann constant. Therefore the total luminosity radiated by a blackbody sphere with radius $R$ is

$$L = 4\pi R^2 f = 4\pi R^2 \sigma T^4.$$  \hfill (104)

Notice that this is also the equation that we used to define the effective temperature of a star. The flux decays like an inverse square law, and hence the flux received by an observer at distance $d$ is

$$f = \frac{L}{4\pi d^2} = \sigma T^4 \left(\frac{R}{d}\right)^2.$$  \hfill (105)

Now if we have a mixture of ideal gas and radiation, then the total pressure is

$$P = P_{gas} + P_r = \frac{\rho}{\mu m_p} kT + \frac{1}{3} aT^4$$  \hfill (106)

At low temperature the gas pressure dominates while at high temperature the radiation pressure dominates. These two contributions are equal when

$$\frac{\rho}{\mu m_p} kT = \frac{1}{3} aT^4$$  \hfill (107)

This happens when

$$\rho \approx \left(\frac{T}{3 \times 10^6}\right)^3 \text{kg m}^{-3}.$$  \hfill (108)

One can easily show that at the centre of the Sun, the gas pressure dominates over the radiation pressure.
3.4.1. Radiation pressure in stars

From eq. (68), we already know that the central temperature is approximately

\[ T_c \approx \frac{GM\mu m_p}{kR} \propto \frac{M}{R} \]  

(109)

while the central gas pressure is

\[ P_c \approx \frac{GM^2}{R^4} \]  

(110)

Now the central radiation pressure is

\[ P_r = \frac{1}{3} a T_c^4 \propto \frac{M^4}{R^4} \]  

(111)

Hence the ratio of the gas pressure to radiation pressure is

\[ \frac{P_r}{P_c} \propto M^2 \]  

(112)

The importance of the radiation pressure increases as the mass of the star increases. Therefore the radiation pressure is much more important in massive stars than in low-mass stars. Loosely speaking, radiation-dominated materials are much more difficult to confine gravitationally. In fact, very massive stars are dominated by the radiation pressure, and stars more massive than 100\( M_\odot \) are known to be unstable to perturbations. Indeed this imposes an upper limit on the mass of stars.

Summary

In a large region in the density - temperature plane, the equation of state and all important thermodynamic quantities may be calculated analytically with a reasonable precision, better than 2%. The region is limited at very low density by the mass density associated with radiation energy density becoming dominant, at low temperature by partial recombination of helium and hydrogen, at high density and low temperature by the crystallisation of ions, and at high temperature by creation of electron-positron pairs. As long as we stay within this very large region, the following prescription can be adopted to calculate equation of state.

Let \( X \), \( Y \), and \( Z \) be the abundances of hydrogen, helium and heavy elements, respectively, all by mass fraction. Of course, we have \( X + Y + Z = 1 \). Various mean molecular weights can be calculated as follows:

\[ \frac{1}{\mu_i} \equiv \frac{n_i m_p}{\rho} = X + \frac{Y}{4} + \frac{Z}{16}, \quad \frac{1}{\mu_e} \equiv \frac{n_e m_p}{\rho} = \frac{1 + X}{2}, \]  

(113)

\[ \frac{1}{\mu} \equiv \frac{n m_p}{\rho} = \frac{1}{\mu_i} + \frac{1}{\mu_e} \approx 2X + 0.75Y + 0.56Z, \quad n = n_i + n_e, \]  

(114)

where \( n \), \( n_i \), and \( n_e \), is the number density of all particles, ions, and electrons, respectively, and \( m_p \) is the mass of a proton.

In the whole region of interest the ion and radiation pressure can be calculated according to

\[ P_i = \frac{\rho}{\mu_i m_p} kT, \quad \frac{k}{m_p} = 0.83 \times 10^4 \quad [\text{J kg}^{-1} \text{K}^{-1}], \]  

(115)
The electron pressure can be calculated according to the following set of formulae:

\[ P_{e,\text{nd}} = \frac{\rho}{\mu_e m_p} kT, \]
\[ P_{e,\text{NR}} = K_{\text{nr}} \rho^{5/3}, \quad K_{\text{nr}} = 0.99 \times 10^7 \mu_e^{-5/3}, \]
\[ P_{e,\text{UR}} = K_{\text{ur}} \rho^{4/3}, \quad K_{\text{ur}} = 1.23 \times 10^{10} \mu_e^{-4/3}, \]
\[ P_{e,d} = \left( P_{e,\text{NR}} + P_{e,\text{UR}} \right)^{-1/2}, \quad P_e = \left( P_{e,\text{nd}}^2 + P_{e,d}^2 \right)^{1/2}. \]
\[ U_i = 1.5 P_i, \quad U_r = 3 P_r. \]

The total pressure is simply a sum of the electron, ion and radiation pressure:

\[ P = P_t + P_i + P_e \]

### 3.5. Ionisation and Saha equation

We already mentioned previously that inside stars most elements are fully ionised. In this section, we will apply statistical mechanics to consider the ionisation of elements inside stars.

As we all know, if we have two energy levels, \( E_1 \) and \( E_2 \), then, in thermal equilibrium, the number of particles in these two levels must satisfy the Boltzmann equation

\[ \frac{N_1}{N_2} = \frac{g_1}{g_2} e^{-(E_1 - E_2)/kT} \]

Now we will apply the Boltzmann equation to the ionisation of elements. Let us consider two ions, the “i”-th and “i+1”-th of the same element. The “i”-th ion means that the atom has already lost \( i \) electrons, and the “i+1”-th ion means that the atom has lost \( i + 1 \) electrons. The ionisation potential, \( \chi \), is the energy needed to ionise the “i”-th to “i+1”-th ion.

Let us label the statistical weight of the elements “i” and “i+1” as \( g_i \) and \( g_{i+1} \), and the number densities of the “i”, “i+1” ions and electrons as \( n_i \), \( n_{i+1} \) and \( n_e \), respectively.

The statistical weight of the “i”-th ion is just \( g_i \). The statistical weight of an ion in the upper ionisation state is \( g_{i+1} \) multiplied by the number of possible states in which a free electron may occupy. The energy of a free electron with momentum \( p \) relative to the ground state of the “i”-th ion is

\[ E = \chi + \frac{p^2}{2m}. \]

The number of states available to the electron from momentum \( p \) to \( p + dp \) is

\[ dn_e(p) = V_e g_e 4\pi p^2 dp / h^3, \]

where \( V_e = 1/n_e \) is the ordinary volume occupied by the electron. The Boltzmann equation for ionisations that yield free electrons from \( p \) to \( p + dp \) is then

\[ \frac{n_{i+1}}{n_i} = \frac{g_{i+1}}{g_i} dn_e(p) e^{-(\chi + p^2 / 2m_e) / kT} = \frac{g_{i+1}}{g_i} V_e \frac{g_e 4\pi p^2 dp}{h^3} e^{-(\chi + p^2 / 2m_e) / kT} \]
All upper states, i.e., including electrons with all momenta, are then obtained by integrating over the electron momentum distribution

$$\frac{n_{i+1}}{n_i} = \frac{g_{i+1}}{g_i} \frac{1}{n_e} \int_0^\infty \frac{g_e 4\pi p^2 dp}{\hbar^3} e^{-(\chi + p^2/2m_e)/kT} = \frac{g_{i+1} g_e}{g_i n_e} \frac{1}{n_e} \left( \frac{2\pi m_e kT}{\hbar^2} \right)^{3/2} e^{-\chi/kT}$$  \hspace{1cm} (126)

The last equation can be rewritten as the Saha equation

$$\frac{n_{i+1} n_e}{n_i} = \frac{g_{i+1} g_e}{g_i} \left( \frac{2\pi m_e kT}{\hbar^2} \right)^{3/2} e^{-\chi/kT}$$  \hspace{1cm} (127)

### 3.5.1. The ionisation of pure hydrogen

Let us consider the simplest ionisation system for a pure hydrogen gas. The system is composed of protons, electrons and neutral hydrogen atoms. Let us denote the number density of electrons, neutral hydrogen and all hydrogen as $n_e$, $n_{H^0}$, and $n_H$. We obviously have

$$n_H = n_{H^0} + n_{H^+}, \quad n_{H^+} = n_e$$  \hspace{1cm} (128)

And the degree of ionisation is defined as

$$x = \frac{n_{H^+}}{n_H}$$  \hspace{1cm} (129)

The gas is fully ionised if $x = 1$ and fully neutral if $x = 0$. With the definition of $x$, we have

$$n_{H^0} = (1 - x)n_H, \quad n_{H^+} = xn_H$$  \hspace{1cm} (130)

The Saha equation is given by

$$\frac{n_{H^+} n_e}{n_H} = \frac{g_{H^+} g_e}{g_{H^0}} \left( \frac{2\pi m_e kT}{\hbar^2} \right)^{3/2} e^{-\chi/kT}$$  \hspace{1cm} (131)

For hydrogen atom, $\chi = 13.6$eV. Now what are the values of $g_{H^+}, g_e, g_{H^0}$? For the ground state of the neutral hydrogen atom, we have two spin states, i.e., the proton can either be aligned or anti-aligned with the electron, so $g_{H^0} = 2$. For $g_{H^+}$ and $g_e$, we notice that for a fixed proton spin ($g_{H^+} = 1$), the electron can have two spins $g_e = 2$ – we have to be careful to avoid double counting. So for a hydrogen atom,

$$\frac{g_{H^+} g_e}{g_{H^0}} = 1$$  \hspace{1cm} (132)

The counting of states needs to be very carefully done for more complex atoms such as helium and heavier elements.

Hence, we have

$$\frac{n_{H^+} n_e}{n_H} = n_H \frac{x^2}{1 - x} = \left( \frac{2\pi m_e kT}{\hbar^2} \right)^{3/2} e^{-\chi/kT}$$  \hspace{1cm} (133)

It is sometimes convenient to rewrite the above equation in terms of pressure. The total pressure is given by

$$P = (n_{H^0} + n_e + n_{H^+})kT = (1 + x)n_H kT$$  \hspace{1cm} (134)

Substituting $n_H = P/[(1 + x)kT]$ into (133) by $n_H$, we obtain

$$\frac{x^2}{1 - x^2} = \frac{kT}{P} \left( \frac{2\pi m_e kT}{\hbar^2} \right)^{3/2} e^{-\chi/kT}$$  \hspace{1cm} (135)
**Example 3.4** Applications to the Sun:

Let us apply the Saha equation to several radii of the Sun. For the photosphere (which is just below the solar surface), we have \( P = 6.83 \times 10^3 \text{N/m}^2 \), \( T \approx 5600 \text{K} \), we obtain \( x \approx 8 \times 10^{-5} \), and so the gas is almost completely neutral.

For a deeper layer, we have \( P = 1.56 \times 10^{11} \text{N/m}^2 \), \( T \approx 7.15 \times 10^5 \text{K} \), we obtain \( x = 0.993 \), and so the gas is almost fully ionised.

**Exercise 3.5** Verify the above example.

In the stellar atmosphere, we have not only hydrogen, helium but also heavier elements (e.g., Mg, Ca, Fe). The ionisation of all these elements obviously all depend on the temperature. In fact, to obtain their ionisation levels, one must solve coupled equations because the (free) electrons are provided and shared by all the elements.

The energy spectra of stars obviously depend on the ionisation levels, since different ionisation levels will show different absorption patterns. Indeed, for a given metallicity, the stellar spectra, depend primarily on the effective temperature. Hence the stellar classification “O, B, A, F, G, K, M, R, N” is also roughly a sequence in the effective temperature.

### 3.5.2 Limitations of Saha equation

The Saha equation is very useful for studying the ionisation states of elements inside stars\(^1\). However, it has a number of limitations, some of which lead to very interesting physics.

- It is clear from the derivation of Saha equation that we have assumed (local) thermal equilibrium and further it only applies when the electrons follow the Maxwell distribution. It does not apply to cases when the electrons are degenerate, such as those found in white dwarfs.

- If we apply the Saha equation to the centre of the Sun, then \( P_c = 2.60 \times 10^{16} \text{N/m}^2 \), \( T_c = 1.6 \times 10^7 \text{K} \). For simplicity, let us take \( X = 1 \), one finds that \( x \approx 0.76 \), this means 24% of the hydrogen is neutral! However, the partial ionisation is misleading because at the central region, the density is so high that the ionisation potential of hydrogen is lowered.

The physics of this so-called “pressure ionisation” is very interesting. When the density is high enough, the hydrogen atoms can no longer be viewed as isolated, instead they touch each other. This can be easily seen as follows:

\[
\rho_c \approx 1.5 \times 10^5 \text{kg m}^{-3},
\]

which implies a number density of hydrogen and mean separation of

\[
n \approx \frac{\rho_c}{m_p} = 0.9 \times 10^{32} \text{kg m}^{-3}, \quad d \approx n^{-1/3} = 2.2 \times 10^{-11} \text{m}
\]

\(^1\)This subsection is optional and will not be examined.
The mean separation is smaller than the Bohr radius $a_0 = 5.3 \times 10^{-11}$ m. Therefore in this case, the atoms touch each other and as a result the electrons are shared among different protons. Because electrons are Fermions, the shared electrons are constrained: their wave functions must satisfy the anti-symmetry. In other words, their phase space is more limited, therefore they are forced to move into higher-energy levels. This is equivalent to say that the ionisation energy is lowered. Because the electrons are globally shared, there is yet no completely satisfactory theory for this pressure ionisation.