Golden Equations (Lectures 14 to 17)

$$\hat{\sigma}_x\hat{\sigma}_y = i\hat{\sigma}_z \quad \text{(etc.)}$$

$$\hat{\sigma}_x^2 = \hat{\sigma}_y^2 = \hat{\sigma}_z^2 = \hat{I}$$

$$\hat{H} = -\,\hat{\boldsymbol{\mu}}\cdot\mathbf{B}$$

(component of Hamiltonian for a particle in a magnetic field).

$$|S=1, M=0\rangle = \frac{|\uparrow\rangle|\downarrow\rangle + |\downarrow\rangle|\uparrow\rangle}{\sqrt{2}} = \frac{|\leftarrow\rangle|\leftarrow\rangle - |\rightarrow\rangle|\rightarrow\rangle}{\sqrt{2}}$$

$$|S=0, M=0\rangle = \frac{|\uparrow\rangle|\downarrow\rangle - |\downarrow\rangle|\uparrow\rangle}{\sqrt{2}} = \frac{|\leftarrow\rangle|\rightarrow\rangle - |\rightarrow\rangle|\leftarrow\rangle}{\sqrt{2}}$$

Problems

Lecture 18

1. Consider the vector $|b\rangle$ represented as $b(x) = 1/(1+x)$ for $x \geq 0$, and as $b(x) = 0$ for $x < 0$. Show that (a) $|b\rangle$ is normalised and therefore an element of the space of 1-dimensional square-integrable functions, $\mathcal{L}^2(\mathbb{R})$; (b) $\hat{x}|b\rangle$ is unnormalisable and therefore not in $\mathcal{L}^2(\mathbb{R})$; (c) $\langle b|x|b\rangle$ is infinite.
2. Show that (a) the delta function $\delta(x) = \delta(-x)$ (i.e. it is a symmetric function); (b) $\delta(ax) = \delta(x)/|a|$.
3. Consider the "theta function" $\theta(x-x')$ which is zero if $x - x' < 0$ and equals 1 if $x - x' \geq 0$. (a) Show that

$$\delta(x-x') = \frac{d}{dx}\theta(x-x').$$

That is, show that the expression on the right has the properties of a delta function, i.e. equal to zero if $x \neq x'$ and integral equal to 1 if the integral range covers $x = x'$.

Lecture 19 Momentum space and position space in one spatial dimension:

1. From the notes:
$$\langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{ipx/\hbar}.$$
Given a Gaussian "wave packet"
$$\langle x|\psi(0)\rangle = \frac{1}{(2\pi\sigma^2)^{1/4}} e^{-x^2/4\sigma^2},$$
find $\langle p|\psi(0)\rangle$ (e.g. by expanding in terms of position states using $\hat{I} = \int_{-\infty}^{\infty} |x\rangle\langle x|\, dx$).
NB: this requires a result from complex analysis:
$$\int_{-\infty}^{\infty} e^{-z^2}\, dz = \sqrt{\pi}$$
where $z = x + iy$ is a complex variable and the integral is along *any* path through the complex plane, as long as the real component x runs from $-\infty$ to ∞ and y is always finite.

2. (Challenge) If the wave packet in the previous question represents a free particle ($V(x) = 0$) with mass m at time $t = 0$,
   (a) Write down the time evolution operator $\hat{U}(t)$ in the momentum-space representation. (Note that momentum states are energy eigenstates for a free particle, which gives $\hat{U}(t)$ a relatively simple form).
   (b) By expanding in terms of the momentum basis, show that
$$\langle x|\psi(t)\rangle = \frac{1}{(2\pi)^{1/4}\left(\sigma + \frac{i\hbar t}{2m\sigma}\right)^{1/2}} \exp\left[\frac{-x^2}{4\left(\sigma^2 + \frac{i\hbar t}{2m}\right)}\right]$$
   (c) Find $\langle x\rangle$, $\langle p\rangle$, and Δp for the wave packet at time t, and show that
$$\Delta x = \sigma\left(1 + \frac{\hbar^2 t^2}{4m^2\sigma^4}\right)^{\frac{1}{2}},$$
   hence show that a Gaussian wave function (i.e. $\langle x|\psi(0)\rangle$) is a minimum uncertainty state: $\Delta x\,\Delta p = \hbar/2$.
   (d) How long will it take for the position uncertainty to increase by a factor of $\sqrt{2}$ for (i) an electron in an atom, (ii) a dust particle with $\sigma = 10^{-5}$m and $m = 10^{-14}$ kg?

Lecture 20 The harmonic oscillator:

1. Show that $[\hat{a},\, \hat{a}^\dagger] = 1$, that $\hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$, and that

$$|n\rangle = \frac{(\hat{a}^\dagger)^n}{\sqrt{n!}}|0\rangle.$$

2. Find the matrix representation for $\hat{p}_x$ in the energy basis, and show by matrix multiplication that $[\hat{x},\, \hat{p}_x] = i\hbar$.
3. Challenge: Show that

$$\langle E\rangle = \frac{\Delta p_x^2 + \langle p_x\rangle^2}{2m} + \frac{1}{2}m\omega(\Delta x^2 + \langle x\rangle^2).$$

By substituting $\hat{x}$ and $\hat{p}$ in terms of $\hat{a}$ and $\hat{a}^\dagger$, show that $\langle x\rangle$ and $\langle p\rangle$ both vanish for the states $|n\rangle$, and that

$$\begin{aligned} \Delta x^2 = \frac{\hbar}{2m\omega}\langle n|\left(\hat{a}+\hat{a}^\dagger\right)^2|n\rangle &= \left(n+\frac{1}{2}\right)\frac{\hbar}{m\omega} \\ \Delta p_x^2 &= \left(n+\frac{1}{2}\right)\hbar\omega m = Em \end{aligned}$$

Note that the ground state is a minimum uncertainty state, consistent with its Gaussian wave function.

Lecture 21 Entanglement & non-locality:

1. A machine is set up to emit pairs of particles in opposite directions, which are detected by a pair of "black box" detectors at the two ends of the lab. Each detector can operate in two modes, 'A' or 'B', specified by the setting of a switch. When the particles reach the detectors, each detector flashes a red or green light. The mode at each end is chosen at random, just before the particles arrive, so after a while you have a large sample of flashes for each possible combination of modes at the two ends. You observe the following trends:
   - If the switches are set to B at both ends, sometimes both detectors flash green.
   - If the switch at one end is set to A, and the switch at the other to B, the two detectors never both flash green.

   As an exercise in pure logic, show that if the two particles cannot communicate with each other after after they leave the central emitter, then these results imply that we must sometimes see both lights flash red when mode A is selected at both ends.
2. A pair of spin-1/2 particles is emitted in opposite directions with combined spin state given by

$$|\psi\rangle = \sqrt{\frac{3}{8}}\left(|\downarrow\rangle|\uparrow\rangle + |\uparrow\rangle|\downarrow\rangle\right) - \frac{1}{2}|\uparrow\rangle|\uparrow\rangle.$$

(This is in fact an entangled state, i.e. it cannot be written as a simple direct product like $|g\rangle|h\rangle$). These particles are detected by a pair of Stern-Gerlach experiments at opposite ends of the lab. The SG experiments can be set either (A) measure spin along the z direction (i.e. S_z) or (B) tilted to measure spin in a certain direction $\mathbf{n}$, where spin up and down along $\mathbf{n}$ are given by:

$$|+n\rangle = \sqrt{\frac{3}{5}}|\uparrow\rangle + \sqrt{\frac{2}{5}}|\downarrow\rangle; \quad |-n\rangle = -\sqrt{\frac{2}{5}}|\uparrow\rangle + \sqrt{\frac{3}{5}}|\downarrow\rangle$$

The experiments are set up so that a spin-up result gives a green light and a spin-down gives a red light. Show that the measurements will show exactly the two trends noted in the previous question, but also will *never* show two red lights in mode A!

HINTS:

Lecture 18: 1: Probably the fastest way to do some of the integrals is using the method of partial fractions, e.g. note that $x^2 = (1+x)^2 - 2x - 1$. *2(a) Note that* $\delta(x)$ *is real as it is the limit of a sequence of real functions. As always,* $\langle x|x'\rangle = \langle x'|x\rangle^*$. *(b) Change the variable in the defining integral. 3 You can either consider* $\theta(x)$ *as the limit of a sequence of functions with a finite gradient instead of a sudden step, or you can go back to the definition of the differential as the limit when* $\delta x \to 0$. *Remember that* $\int_a^b (df/dx)dx = \int_{f(a)}^{f(b)} df$. *Lecture 19: 1 When evaluating integrals like* $I = \int dx\, \exp[iax - x^2]$, *complete the square:* $I = \exp(-a^2/4)\int dx\, \exp[-(x-(ia/2))^2]$, *and change variables to* $z = x - (ia/2)$. *By change of variables from the given result,* $I(a) = \int_{-\infty}^{\infty} dx\, \exp(-ax^2) = \sqrt{\pi/a}$. *2(b) Calculate* $\langle p|\psi(t)\rangle = \langle p|\hat{U}(t)|\psi(0)\rangle$ *and transform back to position space. 2(c)* $\int_{-\infty}^{\infty} dx\, x^{2n} \exp(-ax^2) = (-d/da)^n I(a)$. *Lecture 21: 1. A consequence of the second result is that if we see a green light in mode B, we know we will get a red light in mode A at the other end. A consequence of the "no-communication" assumption is that the outcome for a particle cannot depend on the switch setting at the other end, because the setting had not even been chosen when the particle left the emitter. 2. The chance of getting, for instance, two spin-ups in mode B (i.e. two green lights) will be* $|(\langle +n|\langle +n|)|\psi\rangle|^2$. *You only have to show that such probabilities are finite or zero.*

Answers to Handout 4

Lecture 14

1. (a) In the lectures we saw that for $\mathbf{n} = \cos\phi\,\mathbf{i} + \sin\phi\,\mathbf{j}$ we have

$$\boldsymbol{\sigma}\cdot\mathbf{n} = \begin{pmatrix} 0 & e^{-i\phi} \\ e^{i\phi} & 0 \end{pmatrix}.$$

Hence for $\mathbf{n} = \sin\theta(\cos\phi\,\mathbf{i} + \sin\phi\,\mathbf{j}) + \cos\theta\,\mathbf{k}$,

$$\boldsymbol{\sigma}\cdot\mathbf{n} = \cos\theta\,\hat\sigma_z + \sin\theta \begin{pmatrix} 0 & e^{-i\phi} \\ e^{i\phi} & 0 \end{pmatrix} = \begin{pmatrix} \cos\theta & 0 \\ 0 & -\cos\theta \end{pmatrix} + \begin{pmatrix} 0 & \sin\theta\, e^{-i\phi} \\ \sin\theta\, e^{i\phi} & 0 \end{pmatrix}.$$

$$= \begin{pmatrix} \cos\theta & \sin\theta\, e^{-i\phi} \\ \sin\theta\, e^{i\phi} & -\cos\theta \end{pmatrix}.$$

(b) Solving the eigenvalue equation is a bit redundant as you can guess that the eigenvalues of $\boldsymbol{\sigma}\cdot\mathbf{n}$ are ± 1, but for the record

$$|\boldsymbol{\sigma}\cdot\mathbf{n} - \lambda\hat I| = 0 = (\cos\theta - \lambda)(-\cos\theta - \lambda) - \sin^2\theta\, e^{i(\phi-\phi)} = -\cos^2\theta - \sin^2\theta + \lambda^2 = \lambda^2 - 1.$$

Hence the eigenvalues of $\mathbf{S}\cdot\mathbf{n} = (\hbar/2)\boldsymbol{\sigma}\cdot\mathbf{n}$ are $\pm\hbar/2$. Solving for the eigenvectors we have

$$\begin{pmatrix} \cos\theta \mp 1 & \sin\theta\, e^{-i\phi} \\ \sin\theta\, e^{i\phi} & -\cos\theta \mp 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0,$$

We know that both lines in this equation will give us the same answer, since only the ratio x/y is determined by the eigenvector equation; the absolute values come from normalizing the vector. From the first line, we get

$$\frac{x}{y} = \frac{\sin\theta\, e^{-i\phi}}{\pm 1 - \cos\theta} = \frac{\sin\theta\, e^{-i\phi/2}}{(\pm 1 - \cos\theta)\, e^{i\phi/2}},$$

where in the last step we multiply top and bottom by $e^{i\phi/2}$ to make the phasors look more balanced. Choosing the + sign (corresponding to $\lambda = 1$ i.e. $|{+n}\rangle$), and using double-angle formulae, we have

$$\frac{x}{y} = \frac{2\sin(\theta/2)\,\cos(\theta/2)\, e^{-i\phi/2}}{2\sin^2(\theta/2)\, e^{i\phi/2}} = \frac{\cos(\theta/2)\, e^{-i\phi/2}}{\sin(\theta/2)\, e^{i\phi/2}}$$

as required (we can just identify x and y with the numerator and denominator as this obviously gives a normalised eigenvector). Note that for $|{-n}\rangle$ we get $-\cos^2(\theta/2)$ on the bottom instead.

Consistency check: For $|\pm z\rangle$, $\theta = 0$ and π respectively and ϕ is arbitrary. This gives

$$|{+z}\rangle \underset{S_z}{\longrightarrow} \begin{pmatrix} e^{-i\phi/2} \\ 0 \end{pmatrix}; \quad |{-z}\rangle \underset{S_z}{\longrightarrow} \begin{pmatrix} 0 \\ e^{i\phi/2} \end{pmatrix};$$

which differ from the standard spinors only by phase factors. For $|\pm x\rangle$, $\theta = \pi/2$ and $\phi = 0$ or π:

$$|{+x}\rangle \xrightarrow[S_z]{} \begin{pmatrix} \cos(\pi/4) \\ \sin(\pi/4) \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}; \quad |{-x}\rangle \xrightarrow[S_z]{} \begin{pmatrix} -i\cos(\pi/4) \\ i\sin(\pi/4) \end{pmatrix} = \frac{-i}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix};$$

again we have a phase factor $-i = e^{-i\pi/2}$ in the case of $|{-x}\rangle$.

(c) For spin along the y axis, $\theta = \pi/2$ and $\phi = \pi/2$, i.e.

$$|{+y}\rangle \xrightarrow[S_z]{} \frac{1}{\sqrt{2}} \begin{pmatrix} e^{-i\pi/4} \\ e^{i\pi/4} \end{pmatrix}.$$

Comparing this with the answer to Q11.1(b), $\gamma_+ = -\pi/4$, $\gamma_- = \pi/4$, $(\gamma_+ - \gamma_-) = -\pi/2$, consistent with our previous requirement. The conventional phase choice actually has $\gamma_+ = 0$, giving

$$|{+y}\rangle \xrightarrow[S_z]{} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}.$$

(d)

$$\begin{aligned}
\langle +n|\mathbf{S}|+n\rangle &= (\cos(\theta/2)\, e^{i\phi/2}, \sin(\theta/2)\, e^{-i\phi/2}) \frac{\hbar}{2} [\sigma_x\, \mathbf{i} + \sigma_y\, \mathbf{j} + \sigma_z\, \mathbf{k}] \begin{pmatrix} \cos(\theta/2)\, e^{-i\phi/2} \\ \sin(\theta/2)\, e^{i\phi/2} \end{pmatrix} \\
&= (\cos(\theta/2)\, e^{i\phi/2}, \sin(\theta/2)\, e^{-i\phi/2}) \times \frac{\hbar}{2} \times \\
&\quad \left[\begin{pmatrix} \sin(\theta/2)\, e^{i\phi/2} \\ \cos(\theta/2)\, e^{-i\phi/2} \end{pmatrix} \mathbf{i} + \begin{pmatrix} -i\sin(\theta/2)\, e^{i\phi/2} \\ i\cos(\theta/2)\, e^{-i\phi/2} \end{pmatrix} \mathbf{j} + \begin{pmatrix} \cos(\theta/2)\, e^{-i\phi/2} \\ -\sin(\theta/2)\, e^{i\phi/2} \end{pmatrix} \mathbf{k} \right] \\
&= \frac{\hbar}{2} \left[\cos(\theta/2)\sin(\theta/2) \left(\mathbf{i}(e^{i\phi} + e^{-i\phi}) + \mathbf{j}(-ie^{i\phi} + ie^{-i\phi}) \right) + \mathbf{k}(\cos^2(\theta/2) - \sin^2(\theta/2)) \right] \\
&= \frac{\hbar}{2} [\sin\theta\, (\mathbf{i}\cos\phi + \mathbf{j}\sin\phi) + \mathbf{k}\cos\theta] = \frac{\hbar}{2}\mathbf{n}.
\end{aligned}$$

(e) write $p = ae^{i\alpha}$, $q = be^{i\beta}$ where $a, b, \alpha, \beta \in \mathbb{R}$, and $a, b \geq 0$. To get this into a normalized form, extract a factor of $\sqrt{a^2 + b^2}$:

$$\begin{pmatrix} p \\ q \end{pmatrix} = \sqrt{a^2 + b^2} \begin{pmatrix} a'\, e^{i\alpha} \\ b'\, e^{i\beta} \end{pmatrix}$$

with $a' = a/\sqrt{a^2 + b^2}$ etc. But now $a', b' \leq 1$ and $b'^2 = 1 - a'^2$, so we can write $\theta/2 = \arccos a'$. Note that this puts θ between 0 and π. Then $a' = \cos(\theta/2)$ and $b' = \sin(\theta/2)$, so

$$\begin{pmatrix} p \\ q \end{pmatrix} = \sqrt{a^2 + b^2}\, e^{i(\alpha+\beta)/2} \begin{pmatrix} \cos(\theta/2)\, e^{i(\alpha-\beta)/2} \\ \sin(\theta/2)\, e^{i(\beta-\alpha)/2} \end{pmatrix}.$$

This is the required result, with $\phi = \beta - \alpha = \arg(q/p)$, also note that $\theta/2 = \arctan(b'/a') = \arctan(b/a) = \arctan(|q/p|)$; and $A = \sqrt{a^2 + b^2}\, e^{i(\alpha+\beta)/2}$.

Lecture 15

1. Differentiating with respect to time:

$$\frac{\omega_1}{2}\left(\begin{array}{c} (\dot{d}+i(\omega_0-\omega)d)\,e^{i(\omega_0-\omega)t} \\ (\dot{c}+i(\omega-\omega_0)c)\,e^{i(\omega-\omega_0)t} \end{array}\right) = i\left(\begin{array}{c} \ddot{c} \\ \ddot{d} \end{array}\right).$$

We need to uncouple the c and d equations, which we can do by using the undifferentiated equation to substitute c, d for $\dot{c}$, $\dot{d}$ and vice-versa. This gives:

$$-i\frac{\omega_1^2}{4}\left(\begin{array}{c} c \\ d \end{array}\right) + \left(\begin{array}{c} i(\omega_0-\omega)i\dot{c} \\ i(\omega-\omega_0)i\dot{d} \end{array}\right) = i\left(\begin{array}{c} \ddot{c} \\ \ddot{d} \end{array}\right).$$

For $d(t)$ this becomes

$$\ddot{d} - i(\omega-\omega_0)\dot{d} + d\omega_1^2/4 = 0$$

Try a solution of the form $d(t) = e^{i\Omega t}$. Inserting this into the above equation gives

$$-\Omega^2 + (\omega-\omega_0)\Omega + \omega_1^2/4 = 0$$

Solving the quadratic gives:

$$\Omega = \frac{(\omega-\omega_0) \pm \sqrt{(\omega-\omega_0)^2+\omega_1^2}}{2}$$

The $\pm$ indicates that we have two solutions, Ω_+ and Ω_-, hence

$$\begin{aligned} b(t) &= (Ae^{i\Omega_+ t} + Be^{i\Omega_- t})\,e^{i\omega_0 t/2} \\ &= A\exp\left[i\frac{\omega+\sqrt{(\omega-\omega_0)^2+\omega_1^2}}{2}t\right] + B\exp\left[i\frac{\omega-\sqrt{(\omega-\omega_0)^2+\omega_1^2}}{2}t\right] \end{aligned}$$

One pair of boundary conditions is that $a(0) = c(0) = 1$ and $b(0) = d(0) = 0$. Applying the second gives $B = -A$. We can also use $c(0) = 1$ in our initial equation at $t = 0$:

$$\dot{d}(t=0) = -i\frac{\omega_1}{2}c(0)e^{i0} = -i\frac{\omega_1}{2}.$$

But

$$\dot{d}(t) = i(A\Omega_+ + B\Omega_-) = iA\sqrt{(\omega-\omega_0)^2+\omega_1^2}$$

So

$$A = \frac{-\omega_1/2}{\sqrt{(\omega-\omega_0)^2+\omega_1^2}}$$

$$d(t) = A\left(e^{i\Omega_+ t} - e^{i\Omega_- t}\right) = -i\frac{\omega_1 e^{i(\omega-\omega_0)t/2}}{\sqrt{(\omega-\omega_0)^2+\omega_1^2}}\sin\left(\frac{\sqrt{(\omega-\omega_0)^2+\omega_1^2}}{2}t\right)$$

The phase factors vanish when we take the modulus squared:

$$|d(t)|^2 = \frac{\omega_1^2}{(\omega-\omega_0)^2+\omega_1^2}\sin^2\left(\frac{\sqrt{(\omega-\omega_0)^2+\omega_1^2}}{2}t\right),$$

as required.

Lecture 16

1. We keep the subscript 1 and 2 for operators on the single-particle spaces and for their extension into the direct product space, while the total angular momentum operators will have no numerical subscripts. Per the hint, on the direct product space we should write $\mathbf{S}_1 = \mathbf{S}_1 \otimes I$ and $\mathbf{S}_2 = I \otimes \mathbf{S}_2$. Thus the commutator between total angular momentum components is

$$[S_x, S_y] = [S_{1x} + S_{2x}, S_{1y} + S_{2y}] \equiv [S_{1x} \otimes I + I \otimes S_{2x}, S_{1y} \otimes I + I \otimes S_{2y}]$$

From Q7.1(c), operators on different spaces commute, so this boils down to

$$[S_x, S_y] = [S_{1x} \otimes I, S_{1y} \otimes I] + [I \otimes S_{2x}, I \otimes S_{2y}]$$

This is an example of Q7.1(d), where in the single-particle space we put $\Omega_1 = S_{1x}$, $\Lambda_1 = S_{1y}$, and we know their commutator is $\Gamma_1 = i\hbar S_{1z}$, and similarly on the space for particle 2. Then in the direct product space we have

$$[S_x, S_y] = i\hbar S_{1z} \otimes I + i\hbar I \otimes S_{2z} = i\hbar S_z.$$

The other two commutators, $[S_y, S_z]$ and $[S_z, S_x]$ go through in exactly the same way, permuting x, y, z.

2. **(a)** We have

$$S^2 = (\mathbf{S}_1 + \mathbf{S}_2)^2 = S_1^2 + S_2^2 + 2\mathbf{S}_1 \cdot \mathbf{S}_2$$

As noted in the lectures,

$$2\mathbf{S}_1 \cdot \mathbf{S}_2 = 2\left(S_{1x}S_{2x} + S_{1y}S_{2y} + S_{1z}S_{2z}\right),$$

and we also have $S_\pm = S_x \pm iS_y$ so

$$S_x = (S_+ + S_-)/2; \quad S_y = (S_+ - S_-)/2i$$

so

$$\begin{aligned} 2\mathbf{S}_1 \cdot \mathbf{S}_2 &= \frac{2}{4}\left\{(S_{1+} + S_{1-})(S_{2+} + S_{2-}) - (S_{1+} - S_{1-})(S_{2+} - S_{2-})\right\} + 2S_{1z}S_{2z} \\ &= S_{1+}S_{2-} + S_{1-}S_{2+} + 2S_{1z}S_{2z}. \end{aligned}$$

We work out the matrices in the direct product basis for each of these terms, representing the basis kets in the order given in the question, i.e.

$$|\uparrow,\uparrow\rangle \xrightarrow[\text{product}]{} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}; \quad |\uparrow,\downarrow\rangle \xrightarrow[\text{product}]{} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}; \quad |\downarrow,\uparrow\rangle \xrightarrow[\text{product}]{} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}; \quad |\downarrow,\downarrow\rangle \xrightarrow[\text{product}]{} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

These basis kets are all eigenstates of $S_1^2 + S_2^2$ with the same eigenvalue, i.e.

$$s_1(s_1+1)\hbar^2 + s_2(s_2+1)\hbar^2 = 2\frac{1}{2}\frac{3}{2}\hbar^2 = \frac{3\hbar^2}{2}.$$

(since we are adding two spin-halfs). So the matrix representation in the direct product basis is:

$$S_1^2 + S_2^2 \xrightarrow[\text{product}]{} \frac{3\hbar^2}{2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The basis kets are also eigenstates of $2S_{1z}S_{2z}$ with eigenvalues $2m_1\hbar\, m_2\hbar$, where m_1 or m_2 are $\pm 1/2$. we have

$$2S_{1z}S_{2z} \xrightarrow[\text{product}]{} \frac{\hbar^2}{2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The products of raising and lowering operators act on the basis kets as:

$$S_{1+}S_{2-}|\uparrow,\uparrow\rangle \equiv (S_{1+} \otimes S_{2-})\left(|\uparrow\rangle_1 \otimes |\uparrow\rangle_2\right) = (S_{1+}|\uparrow\rangle_1) \otimes (S_{2-}|\uparrow\rangle_2) = 0 \otimes \left(\sqrt{\frac{3}{4} + \frac{1}{4}}\hbar|\downarrow\rangle\right) = 0$$

and so on. Working through all the options gives zeros nearly everywhere:

$$S_{1+}S_{2-} \xrightarrow[\text{product}]{} \hbar^2 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}; \quad S_{1-}S_{2+} \xrightarrow[\text{product}]{} \hbar^2 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Adding all these terms together gives the required matrix

$$S^2 \xrightarrow[\text{product}]{} \hbar^2 \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$

(b) Solving for the eigenvalues of $S^2/\hbar^2$:

$$\det(S^2/\hbar^2 - \lambda I) = 0 = \begin{vmatrix} 2-\lambda & 0 & 0 & 0 \\ 0 & 1-\lambda & 1 & 0 \\ 0 & 1 & 1-\lambda & 0 \\ 0 & 0 & 0 & 2-\lambda \end{vmatrix} = (2-\lambda) \begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} (2-\lambda)$$

where we use the usual rule for breaking large determinants into sums of smaller ones...in this case there is only one term as there is only one entry on the top and bottom rows. Hence we get

$$0 = (2-\lambda)^2[(1-\lambda)^2 - 1] = (2-\lambda)^2\lambda(\lambda - 2)$$

i.e. the eigenvalues are 2 (three times) and 0 (once)...the familiar triplet and singlet. Thus $S^2 = 2\hbar^2 = 1(1+1)\hbar^2$ and $S^2 = 0$, i.e. the total spin $S = 1$ or 0. It should be obvious by looking at the matrix that the top and bottom states of the ladder are eigenstates of spin-1:

$$S^2|\uparrow,\uparrow\rangle = 2\hbar^2|\uparrow,\uparrow\rangle; \quad S^2|\downarrow,\downarrow\rangle = 2\hbar^2|\downarrow,\downarrow\rangle$$

and that these have $M = 1$ and $M = -1$ respectively, ie. they are $|S, M\rangle = |1, 1\rangle$ and $|1, -1\rangle$. So we have to find the other two eigenkets, which must be orthogonal to those two and hence must be superpositions of the middle two of our original basis states. These are both $M = 0$ states since $S_z = S_{1z} + S_{2z}$ acting on them multiples them by $(1/2 - 1/2)\hbar$ or $(-1/2 + 1/2)\hbar$ i.e. 0. More formally using the language of direct products:

$$\begin{aligned} S_z|\uparrow,\downarrow\rangle &\equiv (S_{1z} \otimes I + I \otimes S_{2z})\,(|\uparrow\rangle_1 \otimes |\downarrow\rangle_2) = (S_{1z}|\uparrow\rangle_1) \otimes |\downarrow\rangle_2 + |\uparrow\rangle_1 \otimes (S_{2z}|\downarrow\rangle_2) \\ &= \frac{\hbar}{2}|\uparrow,\downarrow\rangle - \frac{\hbar}{2}|\uparrow,\downarrow\rangle = 0 \end{aligned}$$

and the same for $|\downarrow,\uparrow\rangle$. The two eigenkets of S^2 and S_z that we're looking for must have different quantum numbers, so as M is the same, S must differ, i.e. they must be $|S = 1, M = 0\rangle$ and $|S = 0, M = 0\rangle$ as stated in the question. So the eigenvector problem reduces to a 2×2 matrix:

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \lambda \begin{pmatrix} a \\ b \end{pmatrix}.$$

Taking $\lambda = 0$ (i.e. $S = 0$) we get $a = -b$, so the normalised eigenket is

$$|0, 0\rangle = \frac{|\uparrow,\downarrow\rangle - |\downarrow,\uparrow\rangle}{\sqrt{2}}.$$

Taking $\lambda = 2$ (i.e. $S = 1$) we get $a + b = 2a$ so $b = a$, so

$$|1, 0\rangle = \frac{|\uparrow,\downarrow\rangle + |\downarrow,\uparrow\rangle}{\sqrt{2}}.$$

(c) Substituting eigenstates of spin-x for eigenstates of spin-z, using the formulae given in the question:

$$\begin{aligned}
|0,0\rangle &= \frac{|\uparrow\rangle|\downarrow\rangle - |\downarrow\rangle|\uparrow\rangle}{\sqrt{2}} \\
&= \frac{1}{(\sqrt{2})^3}\left\{(|\leftarrow\rangle + |\rightarrow\rangle)(|\leftarrow\rangle - |\rightarrow\rangle) - (|\leftarrow\rangle - |\rightarrow\rangle)(|\leftarrow\rangle + |\rightarrow\rangle)\right\} \\
&= \frac{1}{2\sqrt{2}}\left\{2(|\leftarrow\rangle|\rightarrow\rangle) - 2(|\rightarrow\rangle|\leftarrow\rangle)\right\} = \frac{|\leftarrow\rangle|\rightarrow\rangle - |\rightarrow\rangle|\leftarrow\rangle}{\sqrt{2}} \\
|1,0\rangle &= \frac{|\uparrow\rangle|\downarrow\rangle + |\downarrow\rangle|\uparrow\rangle}{\sqrt{2}} \\
&= \frac{1}{(\sqrt{2})^3}\left\{(|\leftarrow\rangle + |\rightarrow\rangle)(|\leftarrow\rangle - |\rightarrow\rangle) + (|\leftarrow\rangle - |\rightarrow\rangle)(|\leftarrow\rangle + |\rightarrow\rangle)\right\} \\
&= \frac{1}{2\sqrt{2}}\left\{2(|\leftarrow\rangle|\leftarrow\rangle) - 2(|\rightarrow\rangle|\rightarrow\rangle)\right\} = \frac{|\leftarrow\rangle|\leftarrow\rangle - |\rightarrow\rangle|\rightarrow\rangle}{\sqrt{2}}
\end{aligned}$$

Challenge: We saw in Q13.2(b) that a spin-1/2 pointing in an arbitrary direction $\pm\mathbf{n}$ (angles θ, ϕ) can be represented by

$$|{+n}\rangle \underset{S_z}{\longrightarrow} \begin{pmatrix} \cos(\theta/2)\, e^{-i\phi/2} \\ \sin(\theta/2)\, e^{i\phi/2} \end{pmatrix}; \quad |{-n}\rangle \underset{S_z}{\longrightarrow} \begin{pmatrix} -\sin(\theta/2)\, e^{-i\phi/2} \\ \cos(\theta/2)\, e^{i\phi/2} \end{pmatrix}.$$

where we choose the phase (in effect, sign) of $|{-n}\rangle$ so that it equals $|{-z}\rangle \equiv |\downarrow\rangle$ when $\theta = \phi = 0$. Expanding out the expression given in the question:

$$\begin{gathered}
|{+n}\rangle|{-n}\rangle - |{-n}\rangle|{+n}\rangle = \\
(\cos(\theta/2)\, e^{-i\phi/2}|\uparrow\rangle + \sin(\theta/2)\, e^{i\phi/2}|\downarrow\rangle)(-\sin(\theta/2)\, e^{-i\phi/2}|\uparrow\rangle + \cos(\theta/2)\, e^{i\phi/2}|\downarrow\rangle) \\
-(-\sin(\theta/2)\, e^{-i\phi/2}|\uparrow\rangle + \cos(\theta/2)\, e^{i\phi/2}|\downarrow\rangle)(\cos(\theta/2)\, e^{-i\phi/2}|\uparrow\rangle + \sin(\theta/2)\, e^{i\phi/2}|\downarrow\rangle) \\
= \cos^2(\theta/2)|\uparrow\rangle|\downarrow\rangle - \sin^2(\theta/2)|\downarrow\rangle|\uparrow\rangle + \sin^2(\theta/2)|\uparrow\rangle|\downarrow\rangle - \cos^2(\theta/2)|\downarrow\rangle|\uparrow\rangle \\
= |\uparrow\rangle|\downarrow\rangle - |\downarrow\rangle|\uparrow\rangle.
\end{gathered}$$

Therefore

$$|0,0\rangle = \frac{|\uparrow\rangle|\downarrow\rangle - |\downarrow\rangle|\uparrow\rangle}{\sqrt{2}} = \frac{|{+n}\rangle|{-n}\rangle - |{-n}\rangle|{+n}\rangle}{\sqrt{2}}.$$

This confirms, as you might hope, that a state with zero total spin also has zero spin along any arbitrary axis.

Lecture 17

1. Evaluating as per the question:

$$\left(J_1^2 + J_2^2 + 2J_{1z}J_{2z} + J_{1+}J_{2-} + J_{1-}J_{2+}\right)|j_1, j_1\rangle|j_2, j_2\rangle =$$

$$\left((j_1(j_1+1)\hbar^2 + j_2(j_2+1)\hbar^2 + 2j_1\hbar\, j_2\hbar + 0 + 0\right)|j_1, j_1\rangle|j_2, j_2\rangle$$

which is just a multiple of the original so this is an eigenstate. Notice that the only components of J^2 that can change one of the direct product basis states are the ones involving raising and lowering operators. By definition a stretched state contains the maximum or minimum j_1, j_2 values, so trying to raise one and lower the other always gives zero.