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Golden Equations (Lectures 9 to 13)

$$\langle A \rangle = \langle \psi | A | \psi \rangle$$

$$\hat{U}(t, t_0) = e^{-i\hat{H}(t-t_0)/\hbar}$$

$$\hat{H}|\psi(t)\rangle = i\hbar \frac{d}{dt}|\psi(t)\rangle$$

$$[\hat{J}_x, \hat{J}_y] = i\hbar\hat{J}_z; \quad [\hat{J}_y, \hat{J}_z] = i\hbar\hat{J}_x; \quad [\hat{J}_z, \hat{J}_x] = i\hbar\hat{J}_y$$

$$\hat{J}_\pm |j, m\rangle = c_\pm(j, m)|j, m \pm 1\rangle$$

Exam Info

The exam on this course has the standard format: a compulsory Q1 with several short parts testing knowledge of the basics, and a choice of two out of three other questions. This course was introduced in 2009 as "Fundamentals of Quantum Mechanics" (a misnomer) so there are is only three direct previous exam papers to practice on. Moreover, the syllabus was revised after the first year, so from the 2009 paper, Q1(b), Q1(d) and most of Q4 do not apply to the current course. To find further examples to practice on, check papers on the old course PHYS 30101 *Quantum Mechanics*, choosing questions which match the topics covered here (see the "Learning Objectives" in the Blue book). These will usually involve Dirac notation or matrices. For instance, from the Jan 2008 paper (available on-line), Q1(b), (c), (d), and Q3 would be fair questions for this course.[1]

[1] Usually, questions on 2nd-year papers are a bit simpler than questions on 3rd-year papers. But (M) courses like this one are more mathematically sophisticated than average for the year.

Examples

Lecture 14

1. Generalise the result given in the lectures by solving for the eigenstates of spin in an arbitrary direction $\mathbf{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$.
   (a) Show that the spin operator in this direction is

   $$\hat{\mathbf{S}}\cdot\mathbf{n} \xrightarrow[S_z]{} \frac{\hbar}{2}\begin{pmatrix} \cos\theta & \sin\theta\, e^{-i\phi} \\ \sin\theta\, e^{i\phi} & -\cos\theta \end{pmatrix}.$$

   (b) Solve for the eigenvector for "spin up" along $\mathbf{n}$: $|S_n = \hbar/2\rangle \equiv |{+n}\rangle$, to show that it can be represented in the S_z basis by

   $$|{+n}\rangle \xrightarrow[S_z]{} \begin{pmatrix} \cos(\theta/2)\, e^{-i\phi/2} \\ \sin(\theta/2)\, e^{i\phi/2} \end{pmatrix}.$$

   Convince yourself that (apart from an overall phase) this is consistent with our previous expressions for $|{+z}\rangle$, $|{-z}\rangle$, $|{+x}\rangle$, and $|{-x}\rangle$.
   (c) Show that for $\mathbf{n} = \mathbf{j} = (0, 1, 0)$ this gives phases consistent with the requirements on $|{\pm y}\rangle$ from Q11.1.
   (d) Show that $\langle{+n}|\mathbf{S}|{+n}\rangle = (\hbar/2)\mathbf{n}$.
   NB: this is a 3-vector equation: it is a compact way of writing 3 equations for the x, y, and z components of $\mathbf{S}$, e.g. the z-component gives $\langle{+n}|S_z|{+n}\rangle = (\hbar/2)\cos\theta$.
   (e) Show the converse of part (b), i.e. that an arbitrary ket, represented by

   $$|r\rangle \xrightarrow[S_z]{} \begin{pmatrix} p \\ q \end{pmatrix}$$

   for any $p, q \in \mathbb{C}$ represents a definite spin state $A|{+n}\rangle$ for some θ, ϕ, where A is an arbitrary complex number (since $|r\rangle$ is not normalized). Find θ and ϕ as functions of p and q.
   [NB: this is a peculiarity of spin-1/2; for any larger value of angular momentum we can construct states which are not eigenstates of $\hat{J}_n$ in any direction $\mathbf{n}$.]

Lecture 15

1. (Challenge): The equation for magnetic resonance derived in the lecture was

   $$\frac{\omega_1}{2}\begin{pmatrix} d\, e^{i(\omega_0-\omega)t} \\ c\, e^{i(\omega-\omega_0)t} \end{pmatrix} = i\begin{pmatrix} \dot{c} \\ \dot{d} \end{pmatrix}.$$

(a) Without assuming that $\omega = \omega_0$, differentiate this equation with respect to time and by substituting from the above equation to get $\dot{c}$ in terms of d, $\dot{d}$ in terms of c, and vice versa, separate the $c(t)$ and $d(t)$ variables into two second-order differential equations.

(b) Try a solution of the form $d = Ae^{i\Omega t}$. Find the two possible expressions for Ω, say Ω_+ and Ω_-. Hence the general expression is $d = Ae^{i\Omega_+ t} + Be^{i\Omega_- t}$.

(c) Given the boundary condition that the state starts with spin up, i.e.

$$|\psi(0)\rangle = |\uparrow\rangle \xrightarrow[S_z]{} \begin{pmatrix} c(0) \\ d(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

find the values of A and B.

(d) Hence show that the probability of finding the state in spin down is given by Rabi's formula:

$$|d(t)|^2 = |\langle\downarrow|\psi(t)\rangle|^2 = \frac{\omega_1^2}{(\omega_0 - \omega)^2 + \omega_1^2} \sin^2\left(\frac{\sqrt{(\omega_0-\omega)^2 + \omega_1^2}}{2}\, t\right).$$

Note that from normalization, $|c|^2 + |d|^2 = 1$.

Lecture 16

1. If $\hat{\mathbf{S}} = \hat{\mathbf{S}}_1 + \hat{\mathbf{S}}_2$ is the total spin operator of a system of two spin-half particles, prove that the components of $\hat{\mathbf{S}}$ satisfy the commutation relations

$$[S_x, S_y] = i\hbar S_z, \quad [S_y, S_z] = i\hbar S_x, \quad [S_z, S_x] = i\hbar S_y$$

given that these hold for the components of $\hat{\mathbf{S}}_1$ and $\hat{\mathbf{S}}_2$.

2. In a system of two spin-1/2 particles,

(a) Show that the matrix for S^2 for in the product basis (basis kets $|\uparrow,\uparrow\rangle$, $|\uparrow,\downarrow\rangle$, $|\downarrow,\uparrow\rangle$, $|\downarrow,\downarrow\rangle$) is

$$S^2 \xrightarrow[\text{product}]{} \hbar^2 \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$

(b) Show that in addition to $|S, M\rangle = |1,1\rangle = |\uparrow,\uparrow\rangle$ and $|1,-1\rangle = |\downarrow,\downarrow\rangle$, we can complete the set of eigenkets of S^2 with

$$|1,0\rangle = \frac{|\uparrow\rangle|\downarrow\rangle + |\downarrow\rangle|\uparrow\rangle}{\sqrt{2}} \quad \text{and} \quad |0,0\rangle = \frac{|\uparrow\rangle|\downarrow\rangle - |\downarrow\rangle|\uparrow\rangle}{\sqrt{2}}.$$

(c) Show that we can express the last two kets in terms of the S_x basis, i.e. $\{|\leftarrow\rangle, |\rightarrow\rangle\}$, as

$$|1,0\rangle = \frac{|\leftarrow\rangle|\leftarrow\rangle - |\rightarrow\rangle|\rightarrow\rangle}{\sqrt{2}} \quad \text{and} \quad |0,0\rangle = \frac{|\leftarrow\rangle|\rightarrow\rangle - |\rightarrow\rangle|\leftarrow\rangle}{\sqrt{2}},$$

by substituting

$$|\uparrow\rangle = \frac{|\leftarrow\rangle + |\rightarrow\rangle}{\sqrt{2}}, \quad |\downarrow\rangle = \frac{|\leftarrow\rangle - |\rightarrow\rangle}{\sqrt{2}}.$$

For a challenge, show that

$$|0,0\rangle = \frac{|{+n}\rangle|{-n}\rangle - |{-n}\rangle|{+n}\rangle}{\sqrt{2}}$$

for any direction $\mathbf{n}$.

Lecture 17

1. For a system of two particles with angular momentum quantum numbers j_1 and j_2, verify that the "stretched state" $|j_1, m_1 = j_1\rangle|j_2, m_2 = j_2\rangle$ is an eigenstate of the total angular momentum operator J^2 with quantum number $j = j_1 + j_2$, by letting $J^2 = J_1^2 + J_2^2 + 2J_{1z}J_{2z} + J_{1+}J_{2-} + J_{1-}J_{2+}$ act on it.

HINTS:

Lecture 14: 1(b). You will need all the double-angle formulae for cos and sine! (e). Write p and q as complex numbers in polar form, i.e. $ae^{i\phi}$. Show that after normalization the amplitudes can be written as $\cos(\theta/2)$ and $\sin(\theta/2)$. Pull out an overall phase factor to find A. Lecture 15: 1. Note that the two solutions for Ω must be matched to meet the boundary conditions (i.e. at $t = 0$) for both d and $\dot{d}$. Lecture 16: 1. Note that $\mathbf{S}_1$ should really be written $\mathbf{S} \otimes I$, while $\mathbf{S}_2 = I \otimes \mathbf{S}$. The answer to Q12.1 all but solves this problem also.

Answers to Handout 3

Lecture 9

1. (a)

$$\hat{H}\psi(\mathbf{x}) = \hat{H}[c_1\phi_1(\mathbf{x})+c_2\phi_2(\mathbf{x})] = c_1\hat{H}\phi_1(\mathbf{x})+c_2\hat{H}\phi_2(\mathbf{x}) = c_1E_1\phi_1(\mathbf{x})+c_2E_2\phi_2(\mathbf{x})$$

(last step because ϕ_1, ϕ_2 are energy eigenfunctions).

(b) $\hat{H}$ and E_1 both have dimension of energy (operators have the same physical dimensions as their eigenvalues). c_1 is a dimensionless complex number ("probability amplitude"). Wave functions $\phi_1(\mathbf{x})$, $\psi(\mathbf{x})$ have their dimensions set by the requirement that probabilities (dimensionless) are space integrals over squared wave functions. So for 1 particle in 3 space dimensions ϕ_1 and ψ have dimensions inverse square-root volume (length to the $-3/2$ power). $\mathbf{x}$ is a position vector, with dimension of length.

(c) Possible values are the eigenvalues of the energy eigenfunctions present in the superposition, i.e. E_1, with probability $|c_1|^2$, and E_2, with prob. $|c_2|^2$.

(d) Prob. particle is within 1 nm of the origin is

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{1\,\mathrm{nm}} |\psi(\mathbf{x})|^2 r^2 dr \sin\theta d\theta d\phi$$

2. (a) If you measure an observable represented by an operator, the possible values are the operator's eigenvalues. One could solve the eigenvalues of the L_z matrix but since it is already diagonal you can just read them off the main diagonal, i.e the values are $\hbar \times (1, 0, -1)$. NB since the L_z matrix is diagonal, you can tell that these matrices have been written in the L_z representation.

(b) The vector representing $L_z = \hbar$ is just

$$|L_z = \hbar\rangle \underset{L_z}{\longrightarrow} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Evaluating the various expectations we have

$$\begin{aligned} \langle L_x\rangle &= \langle L_z = \hbar|\hat{L}_x|L_z = \hbar\rangle = (1,0,0)\frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \\ &= (1,0,0)\frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = 0 \end{aligned}$$

$$
\begin{aligned}
\langle L_x^2\rangle &= (1,0,0)\frac{\hbar^2}{2}\begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix}\begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix}\\
&= (1,0,0)\frac{\hbar^2}{2}\begin{pmatrix}1&0&1\\0&2&0\\1&0&1\end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix}=\frac{\hbar^2}{2}\\
\Delta L_x &= \sqrt{\langle L_x^2\rangle-\langle L_x\rangle^2}=\frac{\hbar}{\sqrt{2}}.
\end{aligned}
$$

(c) Eigenvalues of $\hat{L}_x$:

$$
\begin{vmatrix}-l&\hbar/\sqrt{2}&0\\\hbar/\sqrt{2}&-l&\hbar/\sqrt{2}\\0&\hbar/\sqrt{2}&-l\end{vmatrix}=-l\left((-l)^2-\frac{\hbar^2}{2}\right)-\frac{\hbar}{\sqrt{2}}\left(\frac{-l\hbar}{\sqrt{2}}-0\right)=l(\hbar^2-l^2)=0.
$$

So the eigenvalues are $l=\hbar\times(1,0,-1)$, the same as for L_z. Eigenvectors come from

$$
y\hbar/\sqrt{2}=xl;\quad (x+z)\hbar/\sqrt{2}=yl;\quad y\hbar/\sqrt{2}=zl.
$$

Hence, up to phase factors, the eigenvectors are

$$
|L_x=\hbar\rangle\underset{L_z}{\longrightarrow}\frac{1}{2}\begin{pmatrix}1\\\sqrt{2}\\1\end{pmatrix};\quad |L_x=0\rangle\underset{L_z}{\longrightarrow}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\-1\end{pmatrix};\quad |L_x=-\hbar\rangle\underset{L_z}{\longrightarrow}\frac{1}{2}\begin{pmatrix}1\\-\sqrt{2}\\1\end{pmatrix}.
$$

(d) We have

$$
|L_z=-\hbar\rangle\underset{L_z}{\longrightarrow}\begin{pmatrix}0\\0\\1\end{pmatrix}.
$$

Possible outcomes are eigenvalues of $\hat{L}_x$, i.e. $\hbar,0,-\hbar$.

$$
\begin{aligned}
\text{Prob}(L_x=\hbar) &= \left|\frac{1}{2}\left(1,\sqrt{2},1\right)\begin{pmatrix}0\\0\\1\end{pmatrix}\right|^2=\frac{1}{4}\\
\text{Prob}(L_x=0) &= \left|\frac{1}{\sqrt{2}}(1,0,-1)\begin{pmatrix}0\\0\\1\end{pmatrix}\right|^2=\frac{1}{2}\\
\text{Prob}(L_x=-\hbar) &= \left|\frac{1}{2}\left(1,-\sqrt{2},1\right)\begin{pmatrix}0\\0\\1\end{pmatrix}\right|^2=\frac{1}{4}.
\end{aligned}
$$

(e) We can get the eigenvalues of $\hat{L}_x^2$ without solving the eigenvalue equation: by the usual rule for a function of an operator, they are just the squares of the eigenvalues of $\hat{L}_x$: $\hbar^2$ (for $L_x = \pm\hbar$) and 0 (for $L_x = 0$). Since this is a degenerate operator, a measurement projects the original state onto the eigenspace of the observed value; in this case the space spanned by $|L_x = \hbar\rangle$ and $|L_x = -\hbar\rangle$. The probability is just the sum of the probabilities for the two components of the vector which lie in this eigenspace, i.e.

$$\begin{aligned}
\text{Prob}(L_x^2 = \hbar^2) &= |\langle L_x = \hbar|\psi\rangle|^2 + |\langle L_x = -\hbar|\psi\rangle|^2 \\
&= \left[\frac{1}{2}(1, \sqrt{2}, 1)\begin{pmatrix} 1/2 \\ 1/2 \\ 1/\sqrt{2} \end{pmatrix}\right]^2 + \left[\frac{1}{2}(1, -\sqrt{2}, 1)\begin{pmatrix} 1/2 \\ 1/2 \\ 1/\sqrt{2} \end{pmatrix}\right]^2 \\
&= \frac{1}{16}\left[1 + 2\sqrt{2}\right]^2 + \frac{1}{16}\left[1 + 0\right]^2 = \frac{1 + 4\sqrt{2} + 8 + 1}{16} = \frac{5 + 2\sqrt{2}}{8} \\
&= 0.979
\end{aligned}$$

More formally, we could get the probability by applying the projection operator. This can be constructed explicitly using the butterfly operators:

$$\begin{aligned}
\hat{P}_{L_x^2=\hbar} &= \hat{P}_{L_x=\hbar} + \hat{P}_{L_x=-\hbar} \\
&= |L_x = \hbar\rangle\langle L_x = \hbar| + |L_x = -\hbar\rangle\langle L_x = -\hbar|
\end{aligned}$$

In matrix form this becomes

$$\begin{aligned}
[P_{L_x^2=\hbar}] &= \frac{1}{4}\begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}\left(1, \sqrt{2}, 1\right) + \frac{1}{4}\begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}\left(1, -\sqrt{2}, 1\right) \\
&= \frac{1}{4}\left[\begin{pmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 2 & \sqrt{2} \\ 1 & \sqrt{2} & 1 \end{pmatrix} + \begin{pmatrix} 1 & -\sqrt{2} & 1 \\ -\sqrt{2} & 2 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{pmatrix}\right] \\
&= \frac{1}{2}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}.
\end{aligned}$$

Hence

$$\begin{aligned}
\langle\psi|\hat{P}_{L_x^2=\hbar}|\psi\rangle &= = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{\sqrt{2}}\right)\frac{1}{2}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1/2 \\ 1/2 \\ 1/\sqrt{2} \end{pmatrix} \\
&= \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{\sqrt{2}}\right)\frac{1}{4}\begin{pmatrix} 1 + \sqrt{2} \\ 2 \\ 1 + \sqrt{2} \end{pmatrix} = \frac{5 + 2\sqrt{2}}{8}
\end{aligned}$$

as before. The column vector in the last line is the projected ket; to get the state after measurement we just re-normalise it:

$$|\psi'\rangle = \left(10 + 4\sqrt{2}\right)^{-1/2}\begin{pmatrix} 1 + \sqrt{2} \\ 2 \\ 1 + \sqrt{2} \end{pmatrix}.$$

If we now measure L_z we get $\hbar$ or $-\hbar$ each with probability $\frac{3+2\sqrt{2}}{10+4\sqrt{2}}$, or $L_z = 0$ with probability $\frac{4}{10+4\sqrt{2}}$.

(f) Since the particle has a finite probability for all three possible eigenvalues of L_z, it must be in a state of the form:

$$|\psi\rangle = a|L_z = \hbar\rangle + b|L_z = 0\rangle + c|L_z = -\hbar\rangle$$

with $|a|^2 = 1/4$, $|b|^2 = 1/2$, $|c|^2 = 1/4$. Taking the square root and allowing for a complex phase gives the result quoted in the question. The phases of individual components in a superposition are **NOT** irrelevant. The probability of getting $L_x = 0$, given the representation of $|L_x = 0\rangle$ derived in part (c), is

$$\begin{aligned}
\text{Prob}(L_x = 0) &= \left| \frac{1}{\sqrt{2}}(1, 0, -1)\frac{1}{2}\begin{pmatrix} e^{i\delta_1} \\ \sqrt{2}e^{i\delta_2} \\ e^{i\delta_3} \end{pmatrix} \right|^2 = \frac{1}{8}\left|e^{i\delta_1} - e^{i\delta_3}\right|^2 \\
&= \frac{1}{8}\left(e^{i\delta_1} - e^{i\delta_3}\right)\left(e^{-i\delta_1} - e^{-i\delta_3}\right) \\
&= \frac{1}{8}\left(1 - e^{i(\delta_1 - \delta_3)} - e^{i(\delta_3 - \delta_1)} + 1\right) = \frac{1 - \cos(\delta_1 - \delta_3)}{4} \\
&= \frac{1}{2}\sin^2\left(\frac{\delta_1 - \delta_3}{2}\right)
\end{aligned}$$

The irrelevance of the overall phase shows up here in that the probability depends only on the phase *difference* between the two surviving terms: we can always remove an overall phase from $|\psi\rangle$. But the relative phases between the terms are crucial: here, if $(\delta_1 - \delta_3) = 0$ there is no chance of getting $L_x = 0$, while if $(\delta_1 - \delta_3) = \pi$ then we have a 50% probability.

Lecture 10

1. Maclaurin series:

$$f(x) = f(0) + x\frac{f'(0)}{1!} + x^2\frac{f''(0)}{2!} + \dots$$

$$\frac{d}{dx}\left(1 + \frac{x}{N}\right)^N = N\left(1 + \frac{x}{N}\right)^{N-1} \times \frac{1}{N} = \left(1 + \frac{x}{N}\right)^{N-1}.$$

Clearly the n-times differential, evaluated at $x = 0$, $f^{n\prime}(0) = 1$ for $n < N$, i.e. for all n in the limit.

Thus

$$\lim_{N\to\infty}\left(1 + \frac{x}{N}\right)^N = 1 + x + x^2/2! + x^3/3! + \dots$$

which I hope you recognize as the Maclaurin expansion for e^x.

2. (a) We have

$$H(t)|\psi(t)\rangle = i\hbar\frac{d}{dt}|\psi(t)\rangle$$

so

$$H(t)U(t_0,t)|\psi(t_0)\rangle = i\hbar\frac{d}{dt}U(t_0,t)|\psi(t_0)\rangle.$$

But here $|\psi(t_0)\rangle$ is for a fixed time and so does not depend on t. Moreover the Schrödinger equation gives the time evolution for any starting wavefunction $|\psi(t_0)\rangle$ (as long as it is in the set of physically possible wavefunctions). But if two operators give the same result for all vectors in the relevant space, they are equal:

$$H(t)U(t_0,t) = i\hbar\frac{d}{dt}U(t_0,t),$$

as required.

(b) Going back to basics

$$i\hbar\frac{dU(t_0,t)}{dt_0} = i\hbar\lim_{dt_0\to 0}\frac{U(t_0+dt_0,t)-U(t_0,t)}{dt_0}$$

Using $U(t_0,t) = U(t_0,t_1)U(t_1,t)$, with $t_1 = t_0 + dt_0$, we get

$$i\hbar\frac{dU(t_0,t)}{dt_0} = i\hbar\lim_{dt_0\to 0}\frac{I - U(t_0,t_0+dt_0)}{dt_0}U(t_0+dt,t)$$

But by definition

$$H(t_0) = i\hbar\lim_{dt_0\to 0}\frac{U(t_0,t_0+dt_0)-I}{dt_0},$$

so

$$i\hbar\frac{dU(t_0,t)}{dt_0} = -H(t_0)\lim_{dt_0\to 0}U(t_0+dt_0,t) = -H(t_0)U(t_0,t).$$

(c) Putting $f = t - t_0$, we have

$$\frac{dU(t-t_0)}{dt_0} = \frac{d(t-t_0)}{dt_0}\frac{dU(f)}{df} = -\frac{dU(f)}{df},$$

and similarly $dU(t-t_0)/dt = dU(f)/df$. Hence

$$H(t)U(t-t_0) = i\hbar\frac{dU(f)}{df} = -[-H(t_0)U(t-t_0)] = H(t_0)U(t-t_0).$$

But since U is unitary it has an inverse, so we can post-multiply both sides:

$$H(t)UU^{-1} = H(t) = H(t_0)UU^{-1} = H(t_0),$$

as required.

3. (a) Find eigenstates of the Hamiltonian:

$$\begin{vmatrix} E_0-\lambda & 0 & A \\ 0 & E_1-\lambda & 0 \\ A & 0 & E_0-\lambda \end{vmatrix} = 0 = (E_0-\lambda)(E_1-\lambda)(E_0-\lambda) + A(-(E_1-\lambda)A)$$

$$((E_0-\lambda)^2 - A^2)(E_1-\lambda) = 0$$

i.e. $\lambda = E_1$ or $\lambda = E_0 \pm A$. Solving for the eigenstates:

$$\begin{pmatrix} E_0 & 0 & A \\ 0 & E_1 & 0 \\ A & 0 & E_0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = E_1 \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

The first and last lines give

$$(E_0 - E_1)a = -Ac$$

$$(E_0 - E_1)c = -Aa$$

which can only be satisfied if $a = c = 0$ (unless $E_1 = E_0 \pm A$, i.e. eigenvalues degenerate). Thus for $E = E_1$ the normalised eigenstate is

$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \longleftarrow |\text{soggy}\rangle$$

Thus if the system starts $|\text{soggy}\rangle$,

$$|\psi(t)\rangle = e^{-iE_1 t/\hbar}|\text{soggy}\rangle;$$

apart from a phase factor, the system remains $|\text{soggy}\rangle$.

(b) If

$$\begin{pmatrix} E_0 & 0 & A \\ 0 & E_1 & 0 \\ A & 0 & E_0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = (E_0 \pm A) \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

we require $b = 0$ from the second line and $c = \pm a$ from the first and third, so the other eigenstates are

$$|E_0 + A\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \quad \text{and} \quad |E_0 - A\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

Thus

$$|\text{pop}\rangle = \frac{1}{\sqrt{2}}(|E_0 + A\rangle - |E_0 - A\rangle)$$

so the time evolution is given by

$$|\psi(t)\rangle = e^{-i\hat{H}t/\hbar}|\text{pop}\rangle = \frac{e^{-i(E_0+A)t/\hbar}}{\sqrt{2}}(|E_0 + A\rangle - e^{2iAt/\hbar}|E_0 - A\rangle)$$

i.e. the system oscillates between $|\text{pop}\rangle$ and $|\text{snap}\rangle$.

Lecture 11

1. (a) Stern-Gerlach experiments show that the spin in one direction can be considered to be a superposition (with equal weight) of the '+' and '−' spin states in a second, orthogonal, direction. Originally we called these directions x and z, but we could have called them y and z just as easily. For equal weight superposition the probabilities must be 1/2 for each state, i.e. complex amplitudes have modulus $1/\sqrt{2}$ but they can have arbitrary phases, giving the required result:

$$|{+y}\rangle = \frac{e^{i\gamma_+}}{\sqrt{2}}|{+z}\rangle + \frac{e^{i\gamma_-}}{\sqrt{2}}|{-z}\rangle.$$

In general there are also arbitrary phases $\delta_\pm$ for $|{+x}\rangle$, but we have the freedom to choose the phases of $|{\pm z}\rangle$ to make $\delta_\pm = 0$.

Re-labelling the axes x and y instead of z and x will not change the results, i.e. there must be a 50% probability of getting $|{+y}\rangle$ from a particle in state $|{+x}\rangle$, as required:

$$|\langle{+y}|{+x}\rangle|^2 = \frac{1}{2}.$$

(b) With the conventional phase choice ($\delta_\pm = 0$) we have

$$|{+x}\rangle = \frac{1}{\sqrt{2}}(|{+z}\rangle + |{-z}\rangle)$$

$$\langle{+y}| = \frac{1}{\sqrt{2}}(e^{-i\gamma_+}\langle{+z}| + e^{-i\gamma_-}\langle{-z}|)$$

Notice that i becomes $-i$ going from ket to bra.

$$\begin{aligned}\langle{+y}|{+x}\rangle &= \frac{1}{2}(e^{-i\gamma_+}\langle{+z}|{+z}\rangle + e^{-i\gamma_+}\langle{+z}|{-z}\rangle + e^{-i\gamma_-}\langle{-z}|{+z}\rangle + e^{-i\gamma_-}\langle{-z}|{-z}\rangle)\\ &= \frac{1}{2}(e^{-i\gamma_+} + e^{-i\gamma_-}),\end{aligned}$$

where we used the orthonormality of $|{+z}\rangle$, $|{-z}\rangle$.

$$|\langle{+y}|{+x}\rangle|^2 = \frac{1}{4}(1 + e^{-i\gamma_+ + i\gamma_-} + e^{-i\gamma_- + i\gamma_+} + 1) = \frac{1}{2}(1 + \cos(\gamma_+ - \gamma_-))$$

To make this agree with the 50% probability derived above, we need

$$(\gamma_+ - \gamma_-) = \left(n + \frac{1}{2}\right)\pi$$

But for both the $e^{i\gamma_\pm}$ phase factors to be real (i.e. ± 1), we would have needed both $\gamma_\pm$ to be an integral number of πs. Thus at least one phase factor must have an imaginary component. Now the probability amplitudes, from the definition of $|+y\rangle$, are

$$\langle \pm z|+y\rangle = \exp[-i\gamma_\pm]/\sqrt{2},$$

so if the phase factors are complex, so are the probability amplitudes.

2. (a) $S_z|\uparrow\rangle = (\hbar/2)|\uparrow\rangle$ and $S_z|\downarrow\rangle = (-\hbar/2)|\downarrow\rangle$; now:

$$\frac{\hbar}{2}(P_\uparrow - P_\downarrow)|\uparrow\rangle = \frac{\hbar}{2}(|\uparrow\rangle + 0)$$
$$\frac{\hbar}{2}(P_\uparrow - P_\downarrow)|\downarrow\rangle = \frac{\hbar}{2}(0 - |\downarrow\rangle)$$

as required.

(b) $S_+|\uparrow\rangle = 0$ and $S_+|\downarrow\rangle = \hbar|\uparrow\rangle$; now

$$\hbar|\uparrow\rangle\langle\downarrow|\uparrow\rangle = \hbar|\uparrow\rangle \times 0 = 0$$
$$\hbar|\uparrow\rangle\langle\downarrow|\downarrow\rangle = \hbar|\uparrow\rangle \times 1 = \hbar|\uparrow\rangle$$

as required; similarly for S_- and $\hbar|\uparrow\rangle\langle\downarrow|$.

3. For $j = 2$ the possible m values are $-2, -1, 0, 1, 2$. We have $j(j+1) = 6$. Remembering that the J_+ (J_-) operators are only non-zero for the diagonal above (below) the main diagonal, we get the following values for $c_\pm$:

m	$c_+(2,m)$	$c_-(2,m)$
2	0	$2\hbar$
1	$2\hbar$	$\sqrt{6}\hbar$
0	$\sqrt{6}\hbar$	$\sqrt{6}\hbar$
-1	$\sqrt{6}\hbar$	$2\hbar$
-2	$2\hbar$	0

i.e. the matrices are

$$J_+ = \hbar \begin{pmatrix} 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{6} & 0 & 0 \\ 0 & 0 & 0 & \sqrt{6} & 0 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}; \quad J_- = \hbar \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{6} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{6} & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \end{pmatrix}.$$

Hence

$$J_x = \frac{J_+ + J_-}{2} = \frac{\hbar}{2} \begin{pmatrix} 0 & 2 & 0 & 0 & 0 \\ 2 & 0 & \sqrt{6} & 0 & 0 \\ 0 & \sqrt{6} & 0 & \sqrt{6} & 0 \\ 0 & 0 & \sqrt{6} & 0 & 2 \\ 0 & 0 & 0 & 2 & 0 \end{pmatrix};$$

$$J_y = \frac{J_+ - J_-}{2i} = \frac{\hbar}{2} \begin{pmatrix} 0 & -2i & 0 & 0 & 0 \\ 2i & 0 & -\sqrt{6}i & 0 & 0 \\ 0 & \sqrt{6} & 0 & -\sqrt{6}i & 0 \\ 0 & 0 & \sqrt{6}i & 0 & -2i \\ 0 & 0 & 0 & 2i & 0 \end{pmatrix}.$$

Lecture 12

1. (a) By definition,

$$\left(\hat{\Theta} \otimes \hat{\Lambda}\right) |v\rangle \otimes |w\rangle = \left(\hat{\Theta}|v\rangle\right) \otimes \left(\hat{\Lambda}|w\rangle\right)$$

for any direct product ket. Therefore

$$\left(\hat{\Omega} \otimes \hat{\Gamma}\right) \left(\hat{\Theta} \otimes \hat{\Lambda}\right) |v\rangle \otimes |w\rangle = \left(\hat{\Omega} \otimes \hat{\Gamma}\right) \left(\hat{\Theta}|v\rangle\right) \otimes \left(\hat{\Lambda}|w\rangle\right) = \left(\hat{\Omega}\hat{\Theta}|v\rangle\right) \otimes \left(\hat{\Gamma}\hat{\Lambda}|w\rangle\right)$$

$$= \left((\hat{\Omega}\hat{\Theta}) \otimes (\hat{\Gamma}\hat{\Lambda})\right) |v\rangle \otimes |w\rangle.$$

Now, we know that we can find a basis on $V_{1\otimes 2}$ consisting of direct product kets of the above form, e.g. $\{|v_i\rangle \otimes |w_j\rangle\}$ where $|v_i\rangle$ and $|w_i\rangle$ are basis vectors on V_1 and V_2 respectively. But since direct product operators are linear (just like all operators we are discussing), if two operators give the same result for all basis vectors they will give the same result for all vectors and so will be identical. Therefore

$$\left(\hat{\Omega} \otimes \hat{\Gamma}\right) \left(\hat{\Theta} \otimes \hat{\Lambda}\right) = (\hat{\Omega}\hat{\Theta}) \otimes (\hat{\Gamma}\hat{\Lambda}) \quad \text{QED.}$$

(b) From the linearity of addition of operators (e.g. Q4.1):

$$\begin{aligned} \left(\hat{A} \otimes \hat{I} + \hat{B} \otimes \hat{I}\right) |v\rangle \otimes |w\rangle &= (\hat{A} \otimes \hat{I})|v\rangle \otimes |w\rangle + (\hat{B} \otimes \hat{I})|v\rangle \otimes |w\rangle \\ &= (\hat{A}|v\rangle) \otimes |w\rangle + (\hat{B}|v\rangle) \otimes |w\rangle \end{aligned}$$

Then from the linearity of the vector direct product:

$$\begin{aligned} &= (\hat{A}|v\rangle + \hat{B}|v\rangle) \otimes |w\rangle = (\hat{A} + \hat{B})|v\rangle \otimes (\hat{I}|w\rangle) \\ &= \left((A + B) \otimes \hat{I}\right) |v\rangle \otimes |w\rangle. \end{aligned}$$

Hence $\hat{A} \otimes \hat{I} + B \otimes \hat{I} = (A + B) \otimes \hat{I}$ by the same argument as before. In this way, operator direct products "inherit" linearity from the vector direct product.

(c) By definition of commutator, and using result (a) above:

$$\begin{aligned}
[\hat{\Omega}_1^{1\otimes 2}, \hat{\Lambda}_2^{1\otimes 2}] &= [\hat{\Omega}_1 \otimes \hat{I}, \hat{I} \otimes \hat{\Lambda}_2] \\
&= (\hat{\Omega}_1 \otimes \hat{I})(\hat{I} \otimes \hat{\Lambda}_2) - (\hat{I} \otimes \hat{\Lambda}_2)(\hat{\Omega}_1 \otimes \hat{I}) \\
&= (\hat{\Omega}_1 \hat{I}) \otimes (\hat{I}\hat{\Lambda}_2) - (\hat{I}\hat{\Omega}_1) \otimes (\hat{\Lambda}_2 \hat{I}) = \hat{\Omega}_1 \otimes \hat{\Lambda}_2 - \hat{\Omega}_1 \otimes \hat{\Lambda}_2 = 0
\end{aligned}$$

(d) Using the results (a) and (b) above:

$$\begin{aligned}
[\hat{\Omega}_1^{1\otimes 2}, \hat{\Lambda}_1^{1\otimes 2}] &= [\hat{\Omega}_1 \otimes \hat{I}, \hat{\Lambda}_1 \otimes \hat{I}] \\
&= (\hat{\Omega}_1 \otimes \hat{I})(\hat{\Lambda}_1 \otimes \hat{I}) - (\hat{\Lambda}_1 \otimes \hat{I})(\hat{\Omega}_1 \otimes \hat{I}) \\
&= \hat{\Omega}_1\hat{\Lambda}_1 \otimes \hat{I} - \hat{\Lambda}_1\hat{\Omega}_1 \otimes \hat{I} = (\hat{\Omega}_1\hat{\Lambda}_1 - \hat{\Lambda}_1\hat{\Omega}_1) \otimes \hat{I} \\
&= [\hat{\Omega}_1, \hat{\Lambda}_1] \otimes \hat{I} = \hat{\Gamma}_1 \otimes \hat{I} = \hat{\Gamma}_1^{1\otimes 2}
\end{aligned}$$

(e) Using the result of part (c) that $\hat{\Omega}_1^{1\otimes 2}$ and $\hat{\Omega}_2^{1\otimes 2}$ commute:

$$\begin{aligned}
\left(\hat{\Omega}_1^{1\otimes 2} + \hat{\Omega}_2^{1\otimes 2}\right)^2 &= \left(\hat{\Omega}_1^{1\otimes 2}\right)^2 + 2\hat{\Omega}_1^{1\otimes 2}\hat{\Omega}_2^{1\otimes 2} + \left(\hat{\Omega}_2^{1\otimes 2}\right)^2 \\
&= \left(\hat{\Omega}_1 \otimes \hat{I}\right)^2 + 2(\hat{\Omega}_1 \otimes \hat{I})(\hat{I} \otimes \hat{\Omega}_2) + \left(\hat{I} \otimes \hat{\Omega}_2\right)^2 \\
&= (\hat{\Omega}_1)^2 \otimes \hat{I} + 2\hat{\Omega}_1 \otimes \hat{\Omega}_2 + \hat{I} \otimes (\hat{\Omega}_2)^2 \quad \text{QED.}
\end{aligned}$$