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Golden Equations from Lectures 1 to 4

These "Golden Equations" are intended as a revision aid. They are **not**, of course, a complete summary of the material in the recent lectures, but if you know all these equations, understand the meaning of all the terms, and can see why these equations are particularly significant, you will be on top of the basic ideas.

$$|a\rangle = \sum_i |i\rangle\langle i|a\rangle = \hat{I}|a\rangle$$

$$\left[\langle b|\hat{A}\right](|a\rangle) = \langle b|\left(\hat{A}|a\rangle\right) \equiv \langle b|\hat{A}|a\rangle$$

$$\langle a|\hat{B}|b\rangle = (\langle a|1\rangle,\ \langle a|2\rangle)\begin{pmatrix} \langle 1|\hat{B}|1\rangle & \langle 1|\hat{B}|2\rangle \\ \langle 2|\hat{B}|1\rangle & \langle 2|\hat{B}|2\rangle \end{pmatrix}\begin{pmatrix} \langle 1|b\rangle \\ \langle 2|b\rangle \end{pmatrix} \equiv [a]^{T*}[B][b]$$

Web resources

Handouts and power-point slides used in the lectures are available on Teachweb and from my web page: `www.jb.man.ac.uk/~jpl/PHYS20602/`

Examples (Lectures 5 to 8)

Lecture 5

1. (revision) For each of the Golden Equations above, define the terms in the equations (e.g.: $|a\rangle$ is an arbitrary ket, a.k.a. abstract vector), and write down in words the key idea(s) expressed in the equation.
2. Show that the product of unitary operators is unitary.

3. Theorem 1.8 in the lectures proved that the columns and rows of a unitary matrix are orthonormal vectors. Prove the converse, i.e. that (i) if the columns of an $N \times N$ matrix are orthonormal vectors, so are the rows (and vice-versa) (ii) that any such matrix is unitary.

Lecture 6

1. Let $\hat{U}$ be a unitary operator: (a) if $\hat{A}$ is Hermitian, show that $\hat{U}\hat{A}\hat{U}^\dagger$ and $\hat{U}^\dagger\hat{A}\hat{U}$ are Hermitian; (b) Show that if the kets $\{|a_i\rangle\}$ form an orthonormal basis, so do $\{|b_i\rangle\}$, where $|b_i\rangle = \hat{U}^\dagger|a_i\rangle$. (c) Show that the determinant of $\hat{U}\hat{A}\hat{U}^\dagger$ is the same as for $\hat{A}$.
2. Why are the following not valid subspaces of $V^3(\mathbb{R})$ (i.e. real 3D space)? (a) The positive x axis; (b) the plane $z = 1$.
3. Show that the set of all vectors orthogonal to a given vector $|v\rangle$ in V^N form an $N-1$ dimensional subspace.
4. If $\hat{A}$ is an operator with eigenkets $|a_n\rangle$ and eigenvalues a_n, we can write

$$\hat{A}|a_n\rangle = a_n|a_n\rangle$$

   (a) Write down the corresponding bra equation (don't assume $\hat{A}$ is Hermitian).
   (b) Let $\hat{A}$ be Hermitian. Show that the $a_n \in \mathbb{R}$, and if $a_j \neq a_k$, $|a_j\rangle$ and $|a_k\rangle$ are orthogonal. (apart from the new notation, this should be revision from PHYS2010).
   (c) Let $\hat{A}$ be unitary. Show that the a_n have the form $e^{i\theta_n}$, where $\theta_n \in \mathbb{R}$, and if $a_j \neq a_k$, $|a_j\rangle$ and $|a_k\rangle$ are orthogonal.

Lecture 7

1. Consider the matrices

$$[A] = \begin{pmatrix} 1 & 3 & 1 \\ 0 & 2 & 0 \\ 0 & 1 & 4 \end{pmatrix}, \quad [B] = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 2 \end{pmatrix}.$$

   For each matrix (a) Is it Hermitian? (b) Find the eigenvalues and normalised eigenvectors (c) Are the eigenvectors orthogonal? (d) If the matrix (say $[A]$) is Hermitian, verify that $[U]^\dagger[A][U]$ is diagonal, $[U]$ being its the matrix of eigenvectors.
2. Consider the matrix

$$[R] = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}.$$

   (i) Show that it is unitary. (ii) Show that its eigenvalues are $e^{i\phi}$ and $e^{-i\phi}$. (iii) Find the corresponding eigenvectors; show that they are orthogonal. (iv) Verify that $[U]^\dagger[R][U]$ is diagonal, where U is the matrix of eigenvectors.

Lecture 8

1. An operator on $V^3(\mathbb{C})$ is represented by the matrix

$$[C] = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

(i) Show that its eigenvalues are $c_1 = c_2 = 0$ and $c_3 = 2$. (ii) Show that $|c_3\rangle$ is represented by any vector of the form

$$\frac{e^{i\theta}}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

(iii) show that the $c = 0$ eigenspace contains all vectors of the form

$$\frac{1}{\sqrt{|g|^2 + 2|f|^2}} \begin{pmatrix} f \\ g \\ -f \end{pmatrix},$$

either by feeding $c = 0$ into the equations or by requiring that the $c = 0$ eigenspace be orthogonal to $|c_3\rangle$.

2. By considering their commutator, show that the matrices $[B]$ and $[C]$ from Q7.1 and Q8.1 respectively may be simultaneously diagonalised. Verify that $[C]$ can be diagonalised with the matrix of eigenvectors that you found for $[B]$ in Q8.1.

3. (a) Show that $e^{\hat{A}} \times e^{\hat{B}} = e^{\hat{A}+\hat{B}}$, provided $[\hat{A}, \hat{B}] = 0$. Use only the abstract operator formalism, not the matrix representation (because in general $\hat{A}$ and $\hat{B}$ may have an uncountable infinity of eigenvectors, in which case there is no straightforward matrix equivalent). (b) Hence, show that if $\hat{A}$ is Hermitian, $e^{i\hat{A}}$ is unitary.

Note: Question number m set for lecture n are referenced as Qn.m, e.g. Lecture 1, Q4 is references as Q1.4.

HINTS:

Lecture 5: 3(i) To prove that rows i and k of a matrix $[A]$ are orthogonal, you need to show that $\sum_j (A_{ij})^* A_{kj} = 0$. Lecture 6: 1(b) Show that the $|b_i\rangle$ are all orthonormal and that there are enough of them to make a basis. 1(c) See the proof of Theorem 1.8 in the lectures. 4(b) Consider $\langle a_j|\hat{A}|a_k\rangle$. Work $\hat{A}$ to both left and right and equate the results (this is exactly analogous to the proof for operators on wave functions given in PHYS20101). 4(c) Consider $\langle a_j|\hat{A}^\dagger \hat{A}|a_k\rangle$. Lecture 7 2(iii) remember that coordinates can be complex. Lecture 8 3(a) Use the power series expansion of e^x and compare terms. 3(b) Don't forget to check that both $U^\dagger U = I$ and $UU^\dagger = I$.

Answers to Handout 1

Lecture 1

1.

(a)	$\vert b\rangle + \vert v\rangle = \vert v\rangle$	Hypothesis
	$\vert v\rangle + \vert -v\rangle = \vert 0\rangle$	Group property 4
	$(\vert b\rangle + \vert v\rangle) + \vert -v\rangle = \vert 0\rangle$	Substitution for $\vert v\rangle$
	$\vert b\rangle + (\vert v\rangle + \vert -v\rangle) = \vert 0\rangle$	Group property 2
	$\vert b\rangle + \vert 0\rangle = \vert 0\rangle$	Group property 4
	Therefore $\vert b\rangle = \vert 0\rangle$	Group property 3.
(b)	$(0+1)\vert a\rangle = 0\vert a\rangle + 1\vert a\rangle$	Property 2(d)
	But also $(0+1)\vert a\rangle = 1\vert a\rangle$	Algebra of ordinary numbers
	$1\vert a\rangle = 0\vert a\rangle + 1\vert a\rangle$	substitution for LHS
	$\vert a\rangle = 0\vert a\rangle + \vert a\rangle$	Property 2(b) applied on each side
	Therefore $0\vert a\rangle = \vert 0\rangle$	Result of part (a).
(c)	$\alpha\vert 0\rangle + \alpha\vert b\rangle = \alpha(\vert 0\rangle + \vert b\rangle)$	Property 2(b)
	But $\vert 0\rangle + \vert b\rangle = \vert b\rangle$	Group property 3
	$\alpha\vert 0\rangle + \alpha\vert b\rangle = \alpha\vert b\rangle$	substitution on RHS of first line
	Therefore $\alpha\vert 0\rangle = \vert 0\rangle$	Result of part (a).

2. Below, (1) and (2) are the top-level requirements for vector spaces, i.e. (1) additive group nature and (2) combinations with scalar (here, real) numbers. (1.n) refers to the nth requirement for groups, and 2(a) etc to the specific properties for combining scalars and vectors, listed on p. 5 of Handout 1.

   (i) Real numbers $\mathbb{R}$. (1) under addition: (1.1) closure: adding two reals gives a real (1.2) addition of reals is associative (1.3) the "zero vector" is the usual 0 (1.4) The inverse of x is $-x$ with the usual meaning of minus. Abelian property: addition of reals is commutative. (2) Combination with real scalars ($\mathbb{R}$ again) via multiplication: a real times a real is a real. 2(a) multipication of reals is associative 2(b) one times a real is the same real 2(c) multiplication is distributive over addition for reals 2(d) same thing again.

   (ii) Real (2×2) matrices (actually the following would work for any $(n \times m)$ real or complex matrices): (1) under addition: as above, except that the "zero vector" is the matrix with all elements zero, and the additive "inverse" is the matrix with the sign of all elements reversed. 2: scalar times a matrix is a matrix 2(a) successive multiplication by scalars is equivalent to multiplication by the product of the scalars, as required 2(b) scalar 1 times a matrix leaves it unchanged 2(c) multiplication by scalar is distributive over matrix addition 2(d) scalar addition and matrix addition are compatible as required.

NB: the rules of vector spaces do not allow you to multiply two vectors together to make a new vector. But the main point about matrices is that you can multiply them together, so it is not really very helpful to think of arbitrary matrices (as opposed to column matrices) as abstract vectors!

3. Sets of functions defined on $0 \le x \le L$. At any one x, $f(x)$ is a real number so we expect sets of arbitrary real functions to pass the requirement for a vector space given that real numbers do (see previous question). The one thing to check is the behaviour at $x = 0$ and $x = L$.

   (i) constrained to zero at the ends: adding zeros gives zero, as does multiply by any number, so this is OK. (The set $\{0\}$ containing just one element, namely zero, is the smallest possible group where the group operation "$\cdot$" is addition, and trivially satisfies all the requirements for combination with scalars, so it is also the smallest possible vector space).

   (ii) Periodic $f(x)$, constrained that $f(0) = f(L)$. This is OK because this property is preserved when such functions are added or multiplied by scalars.

   (iii) $f(0) = 4$: obviously fails to be an additive group since adding two such functions gives $(f + g)(0) = f(0) + g(0) = 8$, so the set is not closed under addition (nor under multiplication by scalars).

Lecture 2

1. (i) $(3, 2, 1) - 2(1, 1, 0) - (1, 0, 1) = (0, 0, 0)$. This is the representation of the zero vector; by convention we just call it 0. QED.

   (ii) The vectors are linearly dependent if we can find a, b, c such that

   $$a(1, 1, 0) + b(1, 0, 1) + c(0, 1, 1) = 0.$$

   Then the three components give (1) $a + b = 0$ (2) $a + c = 0$ (3) $b + c = 0$. Substituting from (1) and (2) in (3) gives $-2a = 0$ hence $a = 0$. Then (1) and (2) give $b = c = 0$. Since there is no solution except all coefficients zero, the three vectors are not linearly dependent.

2. By definition an N dimensional space can accomodate at most N linearly independent vectors; for definiteness call one such set $\{|1\rangle, |2\rangle \ldots |N\rangle\}$. Suppose this was not a basis. Then there would be some vector $|v\rangle$ for which there would be no set of coefficients a_i satisfying

   $$|v\rangle = \sum_{i=1}^{N} a_i |i\rangle$$

Therefore, the only solution to

$$a_0|v\rangle + \sum_{i=1}^{N} a_i|i\rangle = 0$$

would be for all the coefficients to be zero. Therefore, $|v\rangle$ would be linearly independent of the set $\{|i\rangle\}_{i=1}^{N}$, so the latter would not be a maximal set, contrary to our initial definition that the space has dimension N. Therefore there is no such vector $|v\rangle$ and the $|i\rangle$ do indeed form a basis.

Lecture 3

1.

$$\langle i|a\rangle = \langle i| \left(\sum_{j=1}^{N} a_j|j\rangle \right),$$

from the definition of coordinates. Using linearity (on the right) to move the bra inside the sum gives

$$\langle i|a\rangle = \sum_{j=1}^{N} a_j\langle i|j\rangle = \sum_{j=1}^{N} a_j\delta_{ij} = a_i.$$

2. Start with $\mathbf{a} = 3\mathbf{i} + 4\mathbf{j}$. Normalise it:

$$\mathbf{i}' = \mathbf{a}/|a| = \mathbf{a}/5 = \frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}$$

The component of $\mathbf{b} = 2\mathbf{i} - 6\mathbf{j}$ parallel to $\mathbf{i}'$ is

$$(\mathbf{b}\cdot\mathbf{i}')\mathbf{i}' = (\mathbf{b}\cdot\mathbf{a}/5)\mathbf{i}' = -\frac{18}{25}(3\mathbf{i} + 4\mathbf{j})$$

The residual perpendicular to $\mathbf{i}'$ is then

$$\mathbf{b} - (\mathbf{b}\cdot\mathbf{i}')\mathbf{i}' = (2\mathbf{i} - 6\mathbf{j}) + \frac{18}{25}(3\mathbf{i} + 4\mathbf{j}) = \frac{2}{25}\left(52\mathbf{i} - 39\mathbf{j}\right)$$

Normalising it gives

$$\mathbf{j}' = \frac{52\mathbf{i} - 39\mathbf{j}}{\sqrt{52^2 + 39^2}} = \frac{52}{65}\mathbf{i} - \frac{39}{65}\mathbf{j}$$

Check:

$$\mathbf{i}'\cdot\mathbf{j}' = \frac{1}{5\times 65}\left(3\times 52 - 4\times 39\right) = 0$$

We could have got another pair of orthonormal vectors by starting with $\mathbf{b}$.

3. The Schwarz inequality becomes an equality if the two vectors $|a\rangle$ and $|b\rangle$ are "parallel", i.e. $|b\rangle = c|a\rangle$ where c is some complex scalar. To avoid unnecessary square root signs we square both sides: Then the LHS squared is

$$\langle a|a\rangle\langle b|b\rangle = \langle a|a\rangle\langle a|c^*c|a\rangle = |c|^2\langle a|a\rangle^2$$

while the square of the RHS is

$$|\langle a|b\rangle|^2 = \langle a|b\rangle\langle b|a\rangle = \langle a|c|a\rangle\langle a|c^*|a\rangle = |c|^2\langle a|a\rangle^2.$$

This is just what you get from the dot product between arrow vectors: equal to the product of their magnitudes only when they are parallel ($\cos\theta = 1$).

4. Squaring the triangle inequality, the LHS is

$$|c|^2 \equiv \langle c|c\rangle = (\langle a| + \langle b|)(|a\rangle + |b\rangle) = \langle a|a\rangle + \langle a|b\rangle + \langle b|a\rangle + \langle b|b\rangle$$

while the RHS os

$$(|a| + |b|)^2 = |a|^2 + 2|a||b| + |b|^2 = \langle a|a\rangle + 2|a||b| + \langle b|b\rangle$$

Cancelling terms that appear on both sides we have to prove that

$$\langle a|b\rangle + \langle b|a\rangle = \langle a|b\rangle + \langle a|b\rangle^* = 2\Re(\langle a|b\rangle) \leq 2|a||b|$$

But the real part of $\langle a|b\rangle$ must be less than or equal to its modulus:

$$\Re\langle a|b\rangle \leq |\langle a|b\rangle| \leq |a||b|$$

where the last inequality is the Schwarz inequality. QED.

Lecture 4

1. The linear operators clearly form an abelian group using the definition of addition specified in the question, with the "zero vector" being the zero operator represented as a matrix with all elements zero and the additive inverse of $\hat{A}$ being $-1 \times \hat{A}$. Multiplication by scalars works just as for the original set of vectors, since operators are defined via their action on kets, so multiplying an operator by a scalar boils down to multiplying objects like $A|v\rangle$, which belong to the original vector space $V(\mathbb{C})$.

2. If $\hat{A}|v\rangle = |Av\rangle$ for all $|v\rangle$, then $\langle w|\hat{A}|v\rangle = \langle w|Av\rangle$ (for all $|w\rangle$). From our original definition, this implies $\langle Av| = \langle v|\hat{A}^\dagger$ and so $\langle Av|w\rangle = \langle v|\hat{A}^\dagger|w\rangle$. But $\langle Av|w\rangle = \langle w|Av\rangle^*$ from the definition of inner products. Hence

$$\langle v|\hat{A}^\dagger|w\rangle = \langle w|\hat{A}|v\rangle^* \quad \text{for all } |v\rangle, |w\rangle.$$

Reversing the argument, the above equation implies $\langle v|\hat{A}^\dagger|w\rangle = \langle Av|w\rangle$ for all $|w\rangle$. That is, considered as a functional, $\langle v|\hat{A}^\dagger$ is equal to $\langle Av|$. But for a finite-dimensional vector space there is a unique ket corresponding to any linear functional (i.e. bra); by construction in this case it is $|Av\rangle = \hat{A}|v\rangle$. Hence, $\langle v|\hat{A}^\dagger$ is the bra corresponding to $\hat{A}|v\rangle$ for all $|v\rangle$, i.e. our original definition.

3. The matrix elements of $\hat{A}$ are $A_{ij} = \langle i|\hat{A}|j\rangle$, while the (column) matrix elements of $|v\rangle$ are $v_i = \langle i|v\rangle$ and the (row) matrix elements of $\langle v|$ are $v_i^* = \langle v|i\rangle$. Hence the matrix elements of $\hat{A}|v\rangle = |Av\rangle$ are

$$\langle i|\hat{A}|v\rangle = \sum_j A_{ij} v_j$$

The matrix elements of $\langle Av| = \langle v|\hat{A}^\dagger$ are $\langle i|\hat{A}|v\rangle^*$ by the usual rule that if $|a\rangle$ is represented as $[a]$ (a column matrix), then $\langle a|$ is represented as $[a]^{T*}$. Indeed, performing the matrix multiplication to the left:

$$[v]^{T*}[A]^\dagger = (\langle v|1\rangle, \langle v|2\rangle, \ldots \langle v|N\rangle) \begin{pmatrix} \langle 1|\hat{A}|1\rangle & \langle 1|\hat{A}|2\rangle & \ldots & \langle 1|\hat{A}|N\rangle \\ \langle 2|\hat{A}|1\rangle & \ddots & & \\ \vdots & & & \\ \langle N|\hat{A}|1\rangle & \ldots & & \langle N|\hat{A}|N\rangle \end{pmatrix}^\dagger$$

gives the ith element of the bra matrix as

$$\left([v]^{T*}[A]^\dagger\right)_i = \sum_j v_j^* A_{ji}^\dagger = \sum_j v_j^* A_{ij}^* = \sum_j \langle j|v\rangle^* \langle i|\hat{A}|j\rangle^*$$

$$= \left(\langle i|\hat{A}\left(\sum_j |j\rangle\langle j|\right)|v\rangle\right)^* = \langle i|\hat{A}\hat{I}|v\rangle^* = \langle i|\hat{A}|v\rangle^*$$

as required.