PHYS 20602 **Handout 1**

Vector Spaces for Quantum Mechanics

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January 30, 2012

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Examples Classes

This course is about doing maths: you can only learn by tackling the problems. The course handouts contain examples to do after each lecture. Try to solve them as we go along — if you don't keep up you will lose track of the course! If you can't see how to get started, look at the hints at the end of this section; if you don't need the hints, you are doing very well.

This module has an examples class scheduled in the Moseley lecture theatre at 4:00–5:00 pm on Mondays. The classes will actually be run every 2 weeks, specifically:

6th February [Week 2]	20th February [Week 4]
5th March [Week 6]	19th March [Week 8]
23rd April [Week 10]	

PLEASE NOTE The problems in this handout are to be done after the lectures. There is a **different** problem sheet for the examples classes, featuring slightly shorter problems than the lecture handouts.

Examples for Lectures 1 to 4

Lecture 1

1. Prove:
   (a) There is only one zero vector. That is, if there is some vector $|b\rangle$ which adds to some other vector $|v\rangle$ to leave it unchanged, $|v\rangle + |b\rangle = |v\rangle$, then $|b\rangle = |0\rangle$.
   (b) $0|a\rangle = |0\rangle$ for any vector $|a\rangle$;
   (c) $\alpha|0\rangle = |0\rangle$ for any scalar α.

   Here $|0\rangle$ stands for the zero vector; this is non-standard and later we will just write 0, but for now I want you to distinguish between the zero vector (which is defined *only* by the rules on page 5) and the ordinary scalar number zero. Part of the point of this is to prove that the two behave enough alike that we won't get into serious trouble by using the same symbol.

   Think of this as a game: the definitions on page 5 of these notes gives the rules and you lose if you make a move that is not sanctioned there (or sanctioned by a previous proof). Make only one step at a time and write down the justification for each from the definitions, e.g.:

   $$|a\rangle - |a\rangle = |a\rangle + |-a\rangle \qquad \text{(definition of minus operation)}$$
   $$|a\rangle + |-a\rangle = |0\rangle \qquad \text{(definition of inverse)}$$

   You may also use the substitution rule, i.e. if $A = B$, then expression B can be substituted for A and vice-versa—this is really the definition of equality.

2. The mathematical idea of a "vector space" includes many things apart from the simple idea of a set of "arrows". Show that (i) the set of real numbers, and (ii) the set of 2×2 real matrices, (with the usual definitions of + and multiplication by a scalar) satisfy the definition of a real vector space, as quoted on page 5 of this handout. In each case write down the element of the set corresponding to the "zero vector".

3. Do the following form a real vector space, according to the mathematical definition? (i) The set of functions $f(x)$ defined where $0 \le x \le L$, with $f(0) = f(L) = 0$. (ii) The set of functions which are periodic in L, i.e. $f(0) = f(L)$ (iii) The set of functions for which $f(0) = 4$.

Lecture 2

1. (i) Show that the following row vectors are linearly dependent: $(1, 1, 0)$, $(1, 0, 1)$, $(3, 2, 1)$. (ii) Show the opposite for $(1, 1, 0)$, $(1, 0, 1)$, $(0, 1, 1)$.

2. Show that for an N-dimensional vector space, V^N, any set of *linearly independent* vectors with the maximum possible number of elements (i.e. N,

from the definition of dimension), forms a basis. That is, show that any vector in V^N can be expressed as a linear combination of the vectors in such a set.

Lecture 3 Inner products:

1. Let $\{|i\rangle\}_{i=1}^N$ be an orthonormal basis. Prove that $\langle i|a\rangle = a_i$, the ith coordinate of $|a\rangle$ in this basis.
2. (a) Use Gram-Schmidt orthogonalization to get an orthonormal basis in 2D space starting with $\mathbf{a} = 3\mathbf{i} + 4\mathbf{j}$ and $\mathbf{b} = 2\mathbf{i} - 6\mathbf{j}$. (b) Can you get two different orthonormal bases in this way? (Answer yes or no and say why).
3. The Schwarz inequality
$$|a||b| \geq |\langle a|b\rangle|$$
was proved in the lectures. When is the equality satisfied? (Compare with the properties of ordinary 'arrow' vectors).
4. Use the Schwarz inequality to prove the 'triangle inequality': If $|c\rangle = |a\rangle + |b\rangle$ then
$$|c| \leq |a| + |b|.$$
(Recall that the norm of a vector $|v\rangle$ is $|v| = \sqrt{\langle v|v\rangle}$.)

Lecture 4 Operators and matrices

1. Show that the set of all linear operators on a complex vector space $V(\mathbb{C})$ is itself a complex vector space, if we define the '+' and 'multiply by scalar' operations for operators via:
$$(\alpha\hat{A} + \beta\hat{B})|v\rangle \equiv \alpha\hat{A}|v\rangle + \beta\hat{B}|v\rangle.$$
2. The definition of the adjoint of an operator $\hat{A}$ given in the lectures is as follows: for any ket $|v\rangle$, label the ket that results when we apply $\hat{A}$ as $|Av\rangle$, i.e.
$$\hat{A}|v\rangle \equiv |Av\rangle.$$
Then the adjoint, $\hat{A}^\dagger$, is the operator which, acts to the left on the corresponding bra, $\langle v|$ to give the bra version of $|Av\rangle$:
$$\langle v|\hat{A}^\dagger = \langle Av|.$$
Show that this definition is equivalent to:
$$\langle v|A^\dagger|w\rangle = \langle w|A|v\rangle^* \quad \text{for all } |v\rangle, |w\rangle.$$
(To show that two definitions are equivalent, you must show that each version implies the other).

3. Let $|Av\rangle \equiv \hat{A}|v\rangle$, and $\langle Av|$ be the bra equivalent of ket $|Av\rangle$. Show that the matrix representing $\hat{A}^\dagger$, acting to the left on the (row) matrix representing $\langle v|$, gives the row matrix representing $\langle Av|$.

HINTS:

Lecture 1: 1(a) Add $|-v\rangle$ to both sides of $|v\rangle + |b\rangle = |v\rangle$, then use commutivity and associativity. 1(b) Consider $(0+1)|a\rangle$. Lecture 2: 2. Assume the contrary and show that this contradicts the definition of dimension. Lecture 4: 2 To show the old definition implies the new one is trivial. To show the reverse, you may assume that there is one and only one bra corresponding to each ket. This is easy to see from the matrix representation of bras and kets for finite-dimensional spaces (it is not always true for infinite-dimensional spaces). Think of bras as functionals acting on kets, and notice that if two function(al)s give the same result for all arguments ($f(x) = g(x)$ for all x) then they are equal.

Some Definitions

Groups

A group is a system $[G, \cdot]$ of a set, G, and an operation, $\cdot$, such that

1. The set is **closed** under $\cdot$, i.e. $a \cdot b \in G$ for any $a, b \in G$.
2. The operation is **associative**, i.e. $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ for any $a, b, c \in G$.
3. There is an **identity element** $e \in G$, such that $a \cdot e = e \cdot a = a$ for all $a \in G$.
4. Every $a \in G$ has an **inverse element** a^{-1} such that $a \cdot a^{-1} = a^{-1} \cdot a = e$.

If the operation is **commutative**, i.e. $a \cdot b = b \cdot a$, then the group is **abelian**.

NB, in this definition, the operation "$\cdot$" and the inverse sign "$^{-1}$" are placeholders for whichever specific operations are appropriate for the group under discussion. For vector spaces (see below) they actually represent vector addition ("+") and negation ("$-$").

Vector Spaces

A complex vector space is a set, written $V(\mathbb{C})$, of elements called **vectors**, such that

1. There is an operation $(+)$ such that $[V(\mathbb{C}), +]$ is an abelian group. The identity element is written as 0 and is known as the **zero vector**; the inverse of vector x is written as $-x$.

   Note that the minus sign here is just part of the "name" of the inverse vector, it is not an operation that combines two vectors. However, for convenience we will define a minus operation $a - b \equiv a + (-b)$.

2. For any complex numbers $\alpha, \beta \in \mathbb{C}$ and vectors $x, y \in V(\mathbb{C})$, products of numbers with vectors such as αx are vectors in $V(\mathbb{C})$ and

   (a) $\alpha(\beta x) = (\alpha \times \beta)x$
   (b) $1\, x = x$
   (c) $\alpha(x + y) = \alpha x + \alpha y$
   (d) $(\alpha + \beta)x = \alpha x + \beta x$

A real vector space $V(\mathbb{R})$ is the same except that α, β in the above must be real numbers ($\in \mathbb{R}$), not complex. Ordinary 3D vectors belong to a real vector space.

In most of this course, instead of representing abstract vectors as symbols such as a or $-a$, we will use the **Dirac notation**, writing them as $|a\rangle$ or $|-a\rangle$ instead. This prevents them being confused with simple numbers.

Syllabus

1. **Finite-dimensional vector spaces** (9 lectures)

   1.1 Preliminaries: sets and groups

   1.2 Abstract vector spaces

   1.3 Linear independence, bases, and dimensions

   1.4 Inner products

   1.5 Linear operators

   1.6 Hermitian and unitary operators

   1.7 Subspaces and direct sums

   1.8 Eigenvalues and eigenvectors

   1.9 Functions of operators

2. **Quantum mechanics and vector spaces** (3 lectures)

   2.1 The physics of quantum mechanics: Stern-Gerlach Experiments

   2.2 Redundant Mathematical structure

   2.3 Time evolution: the Schrödinger equation

   2.4 Example: Spin precession

3. **Angular Momentum** (4 lectures)

   3.1 Angular momentum commutators

   3.2 Eigenvalues and eigenstates of angular momentum

   3.3 Direct product Spaces

   3.4 Orbital angular momentum vs. spin

   3.5 Pauli spin matrices

   3.6 Spin-$\frac{1}{2}$ eigenstates

   3.7 Example: Magnetic resonance

   3.8 Addition of angular momentum

4. **Function spaces** (3 lectures)

   4.1 Functions as vectors

   4.2 Inner product of functions

   4.3 The $\hat{x}$ operator

   4.4 The momementum operator $\hat{p}$

4.5 Bras vs. kets in infinite dimensions

5. The simple harmonic oscillator (2 lectures)

5.1 Creation and annihilation operators

5.2 Matrix representation of SHO

5.3 Representation vs. Equality

5.4 Position-space wave functions of SHO

5.5 The Classical limit

5.6 Time dependence of SHO

6. Composite systems and entanglement (1 lecture)

6.1 Configuration Space

6.2 Entanglement

6.3 Applications of Entanglement

Books

R. Shankar *Principles of Quantum Mechanics, 2nd Ed.,* Plenum, 1994 £59.99.

The first half of this course is based on chapters 1 and 4 of Shankar's book, which is designed for a much longer course. Chapters 10, 11, 12, 14 and 15 cover the rest of this course but not in the order we'll do it! This is an excellent book for self-study if you are prepared to put in the time; it also covers much of the material in our 3rd and 4th years QM courses.

J. S. Townsend *A Modern Approach to Quantum Mechanics,* Addison Wesley 1992 (reprinted by World Scientific 2000).

This covers most of the course, with the major exception that instead of doing the maths first, the approach is strongly based on physics and little attention is paid to careful mathematical definitions. This book will help you get the hang of things if you are not too worried about solid proofs. Look out for the sign error that runs through most of Chapter 3...

C. J. Isham *Lectures on Quantum theory: Mathematical and Structural Foundations*, Imperial College Press/World Scientific, 1995. £14

Useful background reading if you want a clear and slightly more rigourous introduction to vector spaces in QM than this course. But does not cover most of the applications we discuss, such as angular momentum or perturbation theory. (It is good on entanglement).

F. W. Byron and R. W. Fuller *Mathematics of Classical and Quantum Physics* Dover, 1992.

Really a graduate-level maths text but chapters 3–5 cover vector spaces and carefully and clearly. (NB this is a single-volume reprint of a 2-volume set; the relevant chapters are in the original vol 1 which is also available in the JRULM).