1. **2011 Exam Q3(b) & (c)**

   (a) Let $\{|n\rangle\}_{n=0}^{\infty}$ be the usual energy eigenstates of the SHO. A so-called coherent state of a quantum SHO is given by

   $$|\lambda\rangle = A\sum_{n=0}^{\infty}\frac{\lambda^n}{\sqrt{n!}}|n\rangle,$$

   where A is a normalising constant and λ is any complex number. Show that $|\lambda\rangle$ is an eigenstate of the lowering operator $\hat{a}$, with eigenvalue λ. [8 marks]

   (b) Evaluate $\langle\lambda|\hat{a}^\dagger\hat{a}|\lambda\rangle$ and $\langle\lambda|(\hat{a}^\dagger\hat{a})^2|\lambda\rangle$, and hence show that for state $|\lambda\rangle$, the expected value of the energy quantum number is equal to the square of its uncertainty: $\langle n\rangle = (\Delta n)^2$. [10 marks]

   You may assume that $\hat{a}|n\rangle = \sqrt{n}|n-1\rangle$, and that $[\hat{a},\hat{a}^\dagger] = 1$.

2. The following represent 2-particle states where the particles are each in an SHO potential; kets $|0\rangle, |1\rangle$ etc denote the SHO energy states. Are the states entangled or separable?

   (a) $(|0\rangle|1\rangle + |1\rangle|1\rangle)/\sqrt{2}$

   (b) $\{(|3\rangle + |4\rangle)|3\rangle + (|3\rangle + |6\rangle)|6\rangle)\}/2$

1. (a) Since $\hat{a}$ is a linear operator we have

$$\begin{aligned} \hat{a}|\lambda\rangle &= A\sum_{n=0}^{\infty}\frac{\lambda^n}{\sqrt{n!}}\,\hat{a}|n\rangle \\ &= 0 + A\sum_{n=1}^{\infty}\frac{\lambda^n}{\sqrt{n!}}\sqrt{n}|n-1\rangle \\ &= \lambda A\sum_{n=1}^{\infty}\frac{\lambda^{(n-1)}}{\sqrt{(n-1)!}}|n-1\rangle \\ &= \lambda A\sum_{m=0}^{\infty}\frac{\lambda^m}{\sqrt{m!}}|m\rangle \end{aligned}$$

In the second line the zero comes from trying to lower the ground state $|0\rangle$; in the last line we substitute $m = n-1$. But since m is a dummy index, the last line is just $\lambda|\lambda\rangle$ as required.

(b) From the result of part (a),

$$\langle\lambda|\hat{a}^\dagger\hat{a}|\lambda\rangle = \langle\lambda|\lambda^*\lambda|\lambda\rangle = |\lambda|^2\langle\lambda|\lambda\rangle = |\lambda|^2$$

where we exploit the fact that the $|\lambda\rangle$ are normalised (thanks to the A factor).Note that $\hat{a}^\dagger\hat{a} = N$, the number operator, so we have $\langle N\rangle = |\lambda|^2$.
The other matrix element ($\equiv \langle N^2\rangle$) is a bit more tricky, although again we can use $\hat{a}|\lambda\rangle = \lambda|\lambda\rangle$:

$$\begin{aligned} \langle\lambda|(\hat{a}^\dagger\hat{a})^2|\lambda\rangle &= \langle\lambda|\hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{a}|\lambda\rangle = |\lambda|^2\langle\lambda|\hat{a}\hat{a}^\dagger|\lambda\rangle \\ &= |\lambda|^2\langle\lambda|1+\hat{a}^\dagger\hat{a}|\lambda\rangle = |\lambda|^2(1+|\lambda|^2) \end{aligned}$$

In the second line I used the commutator $[\hat{a},\hat{a}^\dagger]$.
Now, the energy quantum number n is just the eigenvalue of the N operator, so its mean is $\langle n\rangle = \langle N\rangle = |\lambda|^2$, and its variance is

$$(\Delta n)^2 = \langle N^2\rangle - \langle N\rangle^2 = |\lambda|^2 + |\lambda|^4 - (|\lambda|^2)^2 = |\lambda|^2$$

which is equal to the mean, as promised.
Notice that this is exactly the relation between mean and variance for a Poisson distribution: in a coherent state, the distribution of the number of quanta found is indeed governed by the Poisson distribution. The wave function for a coherent state is a minimum-uncertainty Gaussian wavepacket that oscillates in the potential with angular frequency ω; hence it is a good model for a nearly-classical state.

2. (a) is separable: both the $|0\rangle$ and $|1\rangle$ kets of the left-hand particle are multipled by the same ket $|1\rangle/\sqrt{2}$ from the right-hand particle: thus we can write the state as $|\alpha\rangle|1\rangle$, where $|\alpha\rangle = (|0\rangle + |1\rangle)\sqrt{2}$.

   (b) is entangled: state $|3\rangle$ of the LH particle is multiplied by $(|3\rangle + |6\rangle)/2$ from the RH particle, but state $|4\rangle$ is multipled only by $|3\rangle/2$ and state $|6\rangle$ is multiplied only by $|6\rangle/2$. From a theorem from Section 1, the energy states are all orthogonal to each other (they are eigenkets of different eigenvalues of a Hermitian operator, namely the Hamiltonian), so there is no possibility that these expressions could turn out to be actually identical.

   This result is blatantly obvious if you look instead at the factors multiplying each ket for the RH particle; I stuck with analysing the first particle just to illustrate that you can do this sort of thing completely mechanically.