PHYS 20602 **Examples Class 1**

The first four questions here complete the set of proofs of key properties of the zero and inverse vectors that were started in Q1 for lecture 1 and Q1 for lecture 2 on the first handout (hereafter Q1.1 & Q2.1). The first one is the hardest. Questions 5 to 7 are 'finger exercises': these are the sort of simple manipulations you need to do all the time in this course.

1. Using the definition of a vector space (including the definition of a group), show that $|-a\rangle$ is unique, i.e. all vectors which add to $|a\rangle$ to give $|0\rangle$ are equal.

   [Hint: assume the contrary, and prove a contradiction.]

2. Similarly, show that $(-1)|a\rangle = |-a\rangle$.

   [You may use the results proved in Q1.1 on the first handout, and also the result of the previous question.]

3. Show that any set of vectors containing the zero vector $|0\rangle$ is linearly dependent.

   [Hint: check the exact wording of the definition of "linearly dependent" in your notes.]

4. Using the definition of an inner product, and the results from Q1 for Lecture 1 on the first handout, prove that $\langle a|0\rangle = \langle 0|a\rangle = 0$ for all $|a\rangle$.

5. $\{|1\rangle, |2\rangle, |3\rangle\}$ are orthonormal vectors. Let

   $$|\psi\rangle = C\left(|1\rangle + 2i|2\rangle + (1+i)|3\rangle\right),$$

   find the constant C if $|\psi\rangle$ is to be normalised.

6. A certain operator $\hat{G}$ acts on vectors in a 3-D vector space and has the following effects on a certain basis set $\{|1\rangle, |2\rangle, |3\rangle\}$:

   $$\begin{aligned} \hat{G}|1\rangle &= |1\rangle + |2\rangle \\ \hat{G}|2\rangle &= -|1\rangle + |2\rangle \\ \hat{G}|3\rangle &= 0 \end{aligned}$$

   Write down the matrix representation of $\hat{G}$ in this basis. Give the matrix representation of the vector $\hat{G}|\psi\rangle$, where $|\psi\rangle$ is the vector from Q. 5.

7. The kets $|a\rangle$ and $|b\rangle$ are represented in a certain orthonormal basis by $\begin{pmatrix} -2 \\ 2i \end{pmatrix}$ and $\begin{pmatrix} 2+3i \\ 2i \end{pmatrix}$ respectively. Find the numerical values of $\langle a|b\rangle$ and $\langle b|a\rangle$, and the norm of $|c\rangle = |a\rangle + |b\rangle$. [2011 exam question worth 6 marks]

1. Suppose $|-a'\rangle$ was another inverse of $|a\rangle$. Then

$\lvert -a'\rangle + \lvert a\rangle = \lvert 0\rangle$	Hypothesis
$(\lvert -a'\rangle + \lvert a\rangle) + \lvert -a\rangle = \lvert -a'\rangle + (\lvert a\rangle + \lvert -a\rangle)$	Group property 2
$\lvert 0\rangle + \lvert -a\rangle = \lvert -a'\rangle + (\lvert a\rangle + \lvert -a\rangle)$	substitution from 1st line on LHS
$\lvert -a\rangle = \lvert -a'\rangle + (\lvert a\rangle + \lvert -a\rangle)$	Group property 3
$\lvert -a\rangle = \lvert -a'\rangle + \lvert 0\rangle$	Group property 4
Therefore $\lvert -a\rangle = \lvert -a'\rangle$	Group property 3.

2.

$(1 + (-1))\lvert a\rangle = 0\lvert a\rangle$	Algebra of ordinary numbers
But $(1 + (-1))\lvert a\rangle = 1\lvert a\rangle + (-1)\lvert a\rangle$	Property 2(d)
$0\lvert a\rangle = 1\lvert a\rangle + (-1)\lvert a\rangle$	substitution on LHS
$\lvert 0\rangle = 1\lvert a\rangle + (-1)\lvert a\rangle$	result of Q1.1(b)
$\lvert 0\rangle = \lvert a\rangle + (-1)\lvert a\rangle$	Property 2(b)
Therefore $(-1)\lvert a\rangle = \lvert -a\rangle$	result of previous Q.

3. A set of vectors $\{|0\rangle, |v\rangle, |w\rangle \ldots\}$ is linearly dependent if and only if

$$a_0|0\rangle + a_1|v\rangle + a_2|w\rangle + \ldots = |0\rangle$$

for some sequence $(a_0, a_1, a_2 \ldots)$ not all zero. This is obviously true for $a_0 = 1$ (or any non-zero value), and all the other coefficients are zero.

4. We know that $|0\rangle = 0|x\rangle$ for any $|x\rangle$ (Result of Q1.1(b)). Therefore $\langle a|0\rangle = \langle a|0|x\rangle = 0\langle a|x\rangle = 0$, whatever $|x\rangle$ is. Also $\langle 0|a\rangle = \langle a|0\rangle^* = 0^* = 0$.

5.

$$\begin{aligned}\langle\psi|\psi\rangle &= C^* \left(\langle 1| - 2i\langle 2| + (1-i)\langle 3|\right) C \left(|1\rangle + 2i|2\rangle + (1+i)|3\rangle\right) \\ &= |C|^2 \left(1\langle 1|1\rangle + 4\langle 2|2\rangle + 2\langle 3|3\rangle\right)\end{aligned}$$

where we omit the terms involving inner products between orthogonal vectors. Since all kets are normalised, all inner products are unity, so we have $1 = 7|C|^2$. This does not fix the complex phase of C, so in general $C = e^{i\theta}/\sqrt{7}$, where $\theta \in \mathbb{R}$, i.e. any real number. Usually we would choose $\theta = 0$, i.e. $C = 1/\sqrt{7}$.

6. Using the rule that the columns of the matrix are the transformed basis vectors we can just write down the answer:

$$\hat{G} \rightarrow \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence

$$\hat{G}|\psi\rangle \rightarrow \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \frac{1}{\sqrt{7}} \begin{pmatrix} 1 \\ 2i \\ 1+i \end{pmatrix} = \frac{1}{\sqrt{7}} \begin{pmatrix} 1-2i \\ 1+2i \\ 0 \end{pmatrix}.$$

7.

$$\langle a|b\rangle = ((-2)^*, (2i)^*) \begin{pmatrix} 2+3i \\ 2i \end{pmatrix} = -(2, 2i) \begin{pmatrix} 2+3i \\ 2i \end{pmatrix} = -(4+6i-4) = -6i;$$

$$\langle b|a\rangle = \langle a|b\rangle^* = 6i.$$

$$|c\rangle = |a\rangle + |b\rangle \rightarrow \begin{pmatrix} -2+2+3i \\ 2i+2i \end{pmatrix} = \begin{pmatrix} 3i \\ 4i \end{pmatrix};$$

hence

$$|c| = \sqrt{\langle c|c\rangle} = \sqrt{(-3i)(3i) + (-4i)(4i)} = \sqrt{9+16} = \sqrt{25} = 5.$$