Outline

- Series Solutions to ODEs
  - Taylor Series
  - Convergence
  - Linear independence

Series Solutions of ODEs

Taylor Series

Expand $f(x)$ around some point x_0 in terms of $(x - x_0)$

$$\begin{aligned} f(x) &= a_0 + a_1(x - x - 0) + a_2(x - x_0)^2 + \dots \\ &= \sum_{n=0}^{\infty} a_n (x - x_0)^n \end{aligned}$$

Use differentiation

$$\begin{aligned} \frac{d}{dx}(x - x_0)^n &= n\,(x - x_0)^{n-1} \\ \frac{d^n}{dx^n}(x - x_0)^n &= n(n-1)(n-2)\dots 1\,(x - x_0) = n! \\ \frac{d^{n+1}}{dx^{n+1}}(x - x_0)^n &= 0 \end{aligned}$$

and note that $(x - x_0)^m|_{x=x_0} = 0$ if $m \leq 1$.

So

$$\begin{aligned}
\left.\frac{d^n f}{dx^n}\right|_{x=x_0} &= 0+\ldots+0+a_n n!+0+\ldots \\
\rightarrow a_n &= \frac{1}{n!} f^{(n)}(x_0) \\
f(x) &= \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(x_0)\,(x-x_0)^n
\end{aligned}$$

Series converges if ratio of successive terms is less than 1. So if

$$\lim_{n\rightarrow\infty}\left|\frac{a_{n+1}x^{n+1}}{a_n x^n}\right| < 1.$$

But note that if $n+1$ term is zero, need to compare ratio of $n+2$ term to n term.

Linear Independence

All powers of x^n are linearly independent and

$$\sum_{n=0}^{\infty} a_n x^n = 0 \text{ for all } x$$

if and only if $a_n = 0$ for all n.
So the Taylor series expansion of any function is unique.
In general a set of functions $f_i(x)$ are linearly independent if

$$\sum_{i=0}^{\infty} a_i f_i = 0 \text{ for all } x$$

if and only if $a_i = 0$ for all i.
Test: Function $f(x)$ and $g(x)$ are linearly independent if

$$W = \begin{vmatrix} f & g \\ \frac{df}{dx} & \frac{dg}{dx} \end{vmatrix} = fg' - gf' \quad \neq \quad 0.$$

W is the Wronskian.

SHM series solution

See the handout *SHM The Hard Way*

Legendre's equation

$$\left(1-x^2\right)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+l(l+1)y \quad = \quad 0$$

where y is defined for $-1 \leq x \leq +1$.
Arises from separation of variables in spherical polar coordinates.

Legendre's polynomials

Solutions to Legendre's equation subject to the boundary condition that $y(x)$ is regular (the function and its derivatives are finite) at $x = \pm 1$ are polynomials of order l, $P_l(x)$ - Legendre's polynomials.

Orthogonality

$$\int_{-1}^{1} P_l(x)P_m(x)d\,x \;=\; \begin{cases} 0 & \text{if } l \neq m \\ \frac{2}{2l+1} & \text{if } l = m \end{cases}$$

Legendre series expansion of a function $f(x)$ on the region $-1 \leq x \leq +1$ is given by

$$f(x) \;=\; \sum_{l=0}^{\infty} c_l P_l(x)$$

Bessel functions

Bessel's equation

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dz} + \left(x^2 - m^2\right) y \quad = \quad 0$$

with the boundary condition, solution finite at $x = 0$. Equation appears for free waves when separating variables in plane or cylindrical polar coordinates.
Solutions $J_m(x)$ - regular Bessel functions. (Series starts at x^m.)