Outline

- Two more examples of PDEs and their solution
  - Laplace's Equation
  - Heat Flow Equation

Laplace's Equation

Electric field: $\underline{E} = -\underline{\nabla}\phi$ where ϕ is the electrostatic potential.
Gauss's law: $\underline{\nabla} \cdot \underline{E} = 0$ - no enclosed charge.
Combine $\nabla^2\phi = 0$ - Laplace's equation
Problem/Example:
Find $\phi(x, y)$ inside the region
$0 \leq x \leq L$, $0 \leq y \leq L$ where
$\phi(0, y) = 0$, $\phi(x, 0) = \phi(x, L) = 0$, and $\phi(L, y) = V$.

Separate variables using $\phi(x, y) = X(x)Y(y)$. Substitute in to PDE, divide by $\phi(x, y)$,

$$\begin{aligned} \frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} &= 0 \\ \frac{1}{X}\frac{d^2X}{dx^2} &= -\frac{1}{Y}\frac{d^2Y}{dy^2} = k^2 \end{aligned}$$

General solution for $Y(y) = A\cos ky + B\sin ky$. Apply boundary conditions,

$$\begin{aligned} Y(0) &= A = 0 \\ Y(L) &= 0 = B\sin kL \rightarrow k = k_n = \frac{n\pi}{L} \text{ with } n = 1, 2, 3, .. \\ Y_n(y) &= \sin\frac{n\pi y}{L}. \end{aligned}$$

Set $B = 1$ for convenience and use the constant from the x solution.

x direction

$$\begin{aligned}\frac{d^2X}{dx^2} &= k^2X \\ \text{Solutions } X(x) &= Ae^{kx} + Be^{-kx} \\ X_n(x) &= A_ne^{n\pi xx/L} + B_ne^{-n\pi x/L}.\end{aligned}$$

So general solution which satisfies the b.c. is the superposition of these solution

$$\phi(x, yy) = \sum_{n=1}^{\infty} \left(A_ne^{n\pi x/L} + B_ne^{-n\pi x/L}\right)\sin\frac{n\pi y}{L}.$$

Impose the x b.c.

$$\phi(0, y) = \sum_{n=1}^{\infty} (A_n + B_n)\sin\frac{n\pi y}{L} = 0 \text{ for } 0 \leq y \leq L.$$

This is just a Fourier sine series for $f(y) = 0$ with $A_n + B_n = 0$ for all n, so $A_n = -B_n$ and the x functions are

$$X_n(x) = A_n\left(e^{k_nx} - e^{-k_nx}\right) = 2A_n\sinh k_nx$$

At $x = L$

$$\phi(L, y) = \sum_{n=1}^{\infty} 2A_n \sinh n\pi \, \sin \frac{n\pi y}{L} = V \text{ for } 0 \leq y \leq L$$

Define new constant $b_n = 2A_n \sinh n\pi$ so

$$\sum_{n=1}^{\infty} b_n \sin \frac{n\pi y}{L} = V$$

which is a Fourier sine series. The coefficients evaluate to

$$b_n = \begin{cases} \frac{4V}{n\pi} & \text{odd } n \\ 0 & \text{even } n \end{cases}$$

$$\phi(x, y) = \frac{4V}{\pi} \sum_{\text{odd}\, n=1}^{\infty} \frac{1}{n \sinh n\pi} \sinh \frac{n\pi x}{L} \sin \frac{n\pi y}{L}$$

The heat-flow equation

$$\nabla^2 T = \frac{1}{D}\frac{\partial T}{\partial t}$$

where T is the temperature and $D = \kappa / C\rho$ the thermal conductivity with κ the thermal conductivity, C the specific heat capacity and ρ the density.
A diffusion equation.