Outline

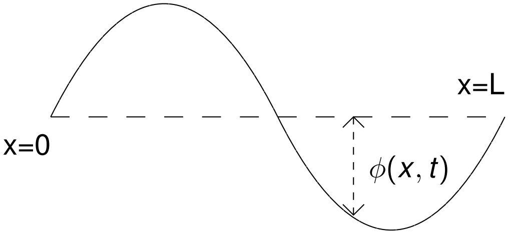
- 1-D Wave Problem: Waves on a string
  - Eigenvalues & Normal Modes
  - Superposition
  - Initial conditions

Wave Problems in 1D

A stretched string

What are these people doing (mathematically) ?

A stretched string



Transverse displacement $\phi(x,t)$ statisfies the *wave equation*

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2}$$

where c is the wave speed.
Boundary conditions ?

The ends of the string are fixed.(At all times.)

$$\phi(0, t) = \quad \phi(L, t) \quad = 0$$

Problem: Given the initial $\phi(x, 0)$, find $\phi(x, t)$.
Example of a *partial differential equation (PDE)*.

Separation of variables

Try a solution of the form

$$\phi(x,t) = X(x)T(t)$$

Differentiate

$$\frac{\partial \phi}{\partial x} = T(t)\frac{dX}{dx}$$
$$\frac{\partial}{\partial x}\frac{\partial \phi}{\partial x} = T\frac{d^2X}{dx^2}$$

Do the same for t. Substitute in to the PDE.

$$\frac{d^2X}{dx^2} = -k^2X \qquad \frac{d^2T}{dt^2} = -c^2k^2T$$

Eigenvalue problems

General solution of SHM

$$X(x) \quad = \quad E \cos kx + F \sin kx$$

Apply boundary conditions:
At $x = 0$, $X(0) = E = 0$
At $x = L$, $X(L) = F \sin kL = 0$
$F = 0$ Or $\sin kL = 0$ which only happens if kL is an integer multiple of π. So for there are a set of values (eigenvalues) k_n such that

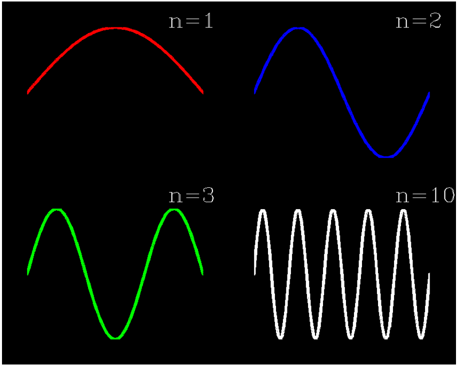
$$k_n L = n\pi, \text{ or } k_n = \frac{n\pi}{L}$$

where $n = 1, 2, 3, ..$ and the eigenfunctions are

$$X_n(x) = F_n \sin \frac{n\pi x}{L}$$

Normal modes

A string oscillates at a set of fixed frequencies - fundamental



and harmonics

- wavelength λ : $\lambda_n = \frac{2L}{n}$ where $n = 1, 2, 3..$
- wavenumber $k = 2\pi/\lambda$: $k_n = \frac{n\pi}{L}$
- angular frequency ω: ($\omega = 2\pi\nu = \frac{2\pi c}{\lambda} = ck$) $\omega_n = \frac{n\pi c}{L}$

Separable solutions

Back to the t variation

$$\begin{aligned} \frac{d^2T}{dt^2} &= -c^2k^2T \\ \rightarrow T(t) &= G\cos ckt + H\sin ckt \\ \text{For } k = k_n,\, T(t) &= G_n\cos\frac{n\pi ct}{L} + H_n\sin\frac{n\pi ct}{L} \end{aligned}$$

Oscillations with angular frequency $w_n = n\pi c/L$ where $n = 1, 2, 3..$

$$\begin{aligned} \phi_n(x,t) &= X_n(x)T(t) \\ &= \sin\frac{n\pi x}{L}\left(A_n\cos\frac{n\pi ct}{L} + B_n\sin\frac{n\pi ct}{L}\right) \end{aligned}$$

Superposition

Wave equation is linear in derivatives of the field

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2}$$

Use *principle of superposition*:
If ϕ_1 and ϕ_2 are each solutions, then so is $\phi_1 + \phi_2$.
The general solution is therefore

$$\begin{aligned} \phi(x,t) &= \sum_{n=1}^{\infty} X_n(x) T_n(t) \\ &= X_1(x)T_1(t) + X_2(x)T_2(t) + ... \end{aligned}$$

For the wave eqn. with $\phi(0,t) = \phi(L,t) = 0$

$$\phi(x,t) = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{L}\left(A_n \cos\frac{n\pi ct}{L} + B_n \sin\frac{n\pi ct}{L}\right)$$

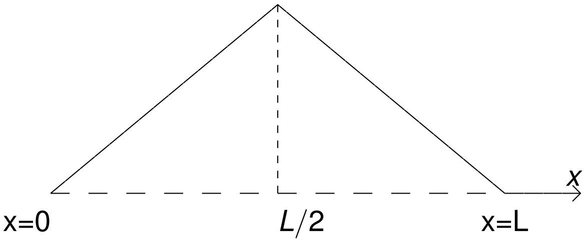
Initial conditions

Want solution for specific starting point.

Example: Pluck string at centre and release from rest.

At time $t = 0$, the displacement is

$$\phi(x,0) = f(x) = \begin{cases} x & \text{for } 0 \leq x \leq \frac{L}{2} \\ L - x & \text{for } \frac{L}{2} < x \leq L \end{cases}$$



Initial velocity = 0, so

$$\left.\frac{\partial \phi}{\partial t}\right|_{t=0} = 0$$

Initial conditions, cont.

At $t = 0$, $\cos 0 = 1$ so $B_n = 0$ but $\sin 0 = 0$, so $A_n \neq 0$ and

$$\phi(x, 0) = A_1 \sin \frac{\pi x}{L} + A_2 \sin \frac{2\pi x}{L} + ...$$

How do we calculate the A_n so that this is equal to

$$f(x) = \begin{cases} x & \text{for } 0 \leq x \leq \frac{L}{2} \\ L - x & \text{for } \frac{L}{2} < x \leq L \end{cases} ?$$

Need a tool to find the constants after applying boundary conditions.
Need to find A_n in

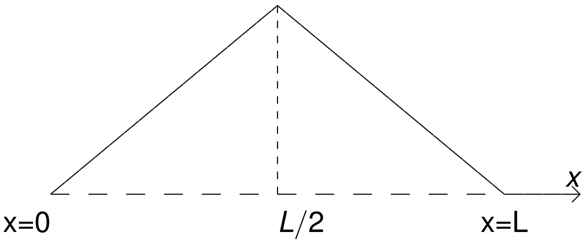
$$\phi(x,0) = A_1 \sin \frac{\pi x}{L} + A_2 \sin \frac{2\pi x}{L} + ...$$

so that $\phi(x,0)$ is a given function. But this is just a Fourier sin series. So

$$A_n \frac{L}{2} = \int_0^L \sin \frac{nx\pi}{L} \phi(x,0)\, dx$$

for $n = 1, 2, 3, ..$

Answer:

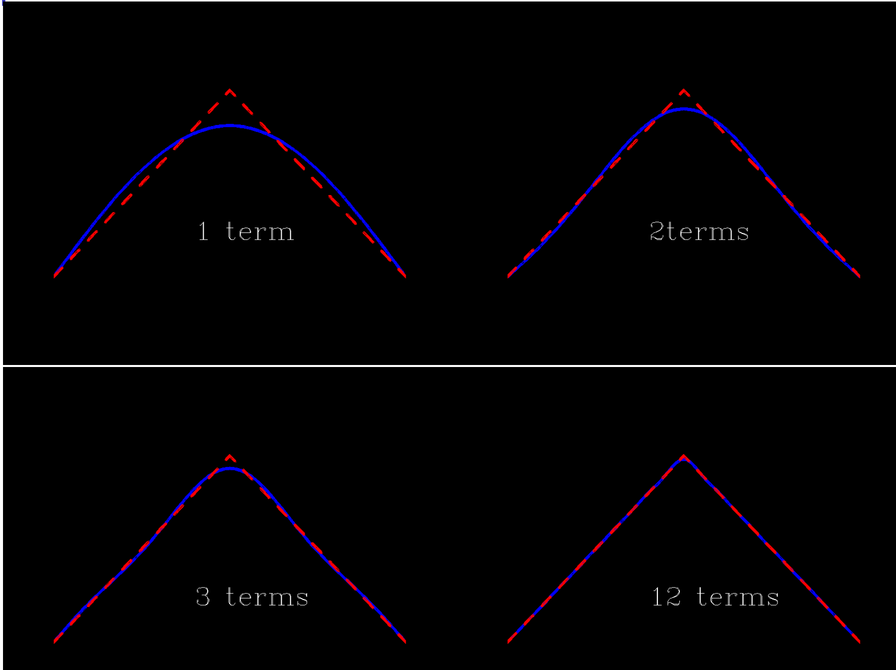


At time $t = 0$, the displacement is

$$f(x) = \begin{cases} x & \text{for } 0 \le x \le \frac{L}{2} \\ L - x & \text{for } \frac{L}{2} < x \le L \end{cases}$$

$$\begin{aligned} f(x) &= \frac{4L}{\pi^2}\left[\sin\left(\frac{\pi x}{L}\right) - \frac{1}{9}\sin\left(\frac{3\pi x}{L}\right) + \frac{1}{25}\sin\left(\frac{5\pi x}{L}\right) + \dots\right] \\ &= \frac{4L}{\pi^2}\sum_{n=\text{odd}}^{\infty}\frac{(-1)^{(n-1)/2}}{n^2}\sin\frac{n\pi x}{L} \end{aligned}$$

A quick look at these terms



Plucked string: Putting it all together

General solution

$$\phi(x,t) = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{L}\left(A_n\cos\frac{n\pi ct}{L} + B_n\sin\frac{n\pi ct}{L}\right)$$

Need to find A_n and B_m given initial displacement:

$$\phi(x,0) = \begin{cases} x & 0 \le x \le L/2 \\ L-x & L/2 \le x \le L \end{cases}$$

and initial velocity

$$\left.\frac{\partial\phi}{\partial t}\right|_{t=0} = 0$$

Note evaluated at $t = 0$ at ALL x.

General solution at $t = 0$

$$\begin{aligned}
\phi(x,0) &= \sum_{n=1}^{\infty} A_n \sin\frac{\pi n x}{L} \\
\to A_n &= \begin{cases} (-1)^{(n-1)/2}\frac{4L}{n^2\pi^2} & > \text{odd } n \\ 0 & \text{even } n \end{cases}
\end{aligned}$$

Also

$$\begin{aligned}
\left.\frac{\partial\phi}{\partial t}\right|_{t=0} &= \sum_{n=1}^{\infty} \sin\frac{n\pi x}{L} B_n \frac{n\pi c}{L} = 0 \\
\to B_n &= 0
\end{aligned}$$

for all n. Hence the solution is

$$\phi(x,t) = \sum_{\text{odd } n=1}^{\infty} (-1)^{(n-1)/2}\frac{4L}{n^2\pi^2} \sin\frac{\pi n x}{L} \cos\frac{n\pi c t}{L}$$

The Method for Solving PDEs

Define problem:

1. PDE
2. Boundary conditions (BC).
3. Initial conditions (IC).

Method:

1. Assume a seperable solution, e.g. $\phi(x, t) = X(x)T(t)$.
2. Separate PDE into a set of ODEs. Introduce separation constants.
3. Solve the ODE + BC eigenvalue problems - normal modes/eigenvectors.
4. Use principle of superposition to write down general solution as sum over the eigenvectors.
5. Apply IC. Use orthogonality of eigenvectors to determine constants in general solution.