Outline

- Waves and Wave Packets

Wave Equation

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2}$$

where c is a constant.
Trial solution on $-\infty < x < \infty$

$$\phi(x,t) = e^{i(kx-\omega t)} = e^{ikx}e^{-i\omega t}$$

$k^2 = \omega^2/c^2$ so $\omega(k) = ck$ and phase velocity $v_p = \omega(k)/k = c$.

Travelling waves

A plane wave of wavenumber k and angular frequency ω is written as

$$e^{i(kx-\omega t)}$$

The phase of the wave at x and t is $kx - \omega t$

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General wave packet (travelling to right)

$$\phi(x,t) = \int_{-\infty}^{\infty} g(k)e^{ik(x-ct)}\,dk$$

Wave Packets

Wave pulse/packet: wave of finite spatial/temporal extent.
Typical wavenumber k_0, spatial width of packet Δx.
In time domain this corresponds to a typical angular frequency ω_0, finite duration Δt.
Fourier transform to look at amplitudes of component angular frequencies

$$g(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega t} f(t)\, dt$$

Bandwidth Theorem

For any function $f(x)$ and its Fourier transform $g(k)$, the widths of the functions Δx and Δk respectively satisfy

$$\Delta x \, \Delta k \geq \frac{1}{2}$$

(In the time domain

$$\Delta t \, \Delta \omega \geq \frac{1}{2}$$

)

Origin of Heisenberg's uncertainty principle in quantum mechanics.

Group Velocity

Wave equations can be dispersive: The phase velocity depends on k. A wave packet spreads out. General wave (moving to right):

$$\begin{aligned} f(x,t) &= \int_{-\infty}^{\infty} g(k) e^{i(kx-\omega(k)t)}\, dk \\ f(x,t) &\approx e^{i(k_0 x-\omega(k_0)t)} \int_{-\infty}^{\infty} g(k_0+q) e^{iq(x-\omega' t)} dq \end{aligned}$$

Group velocity

$$v_g = \left| \frac{d\omega}{dk} \right| = |\omega'|$$

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