Outline

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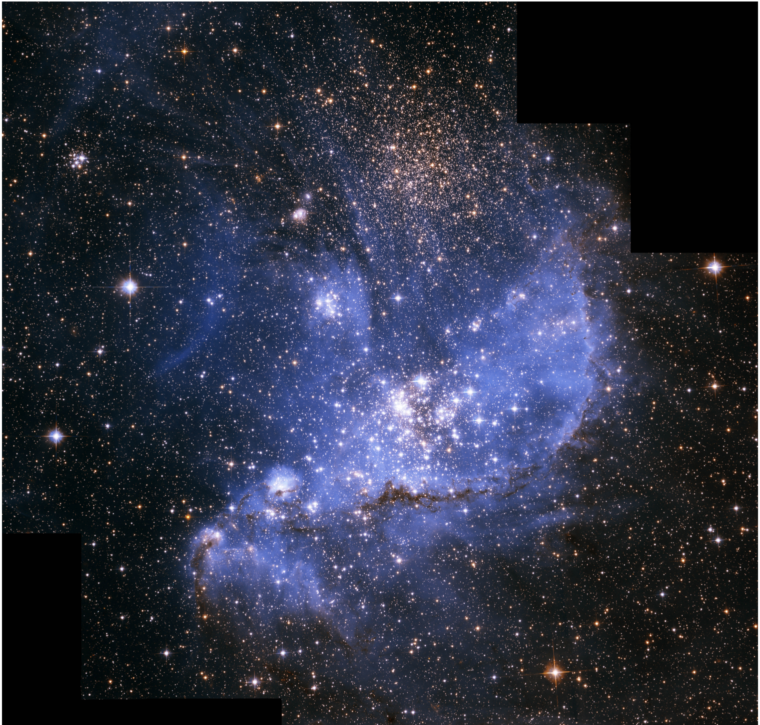
  - Convolution Theorem



Convolution

Definition: The convolution of two functions $f_1(x)$ and $f_2(x)$ is defined as

$$F(x) = \int_{-\infty}^{\infty} f_1(x-u) f_2(u)\, du$$

Symmetric in f_1 and f_2.
Example:

Convolution Example 1

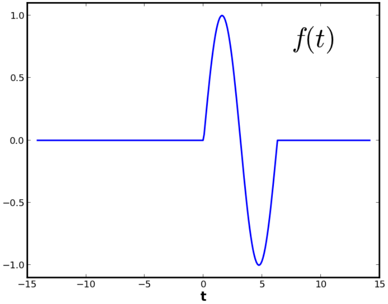
Calculate the convolution

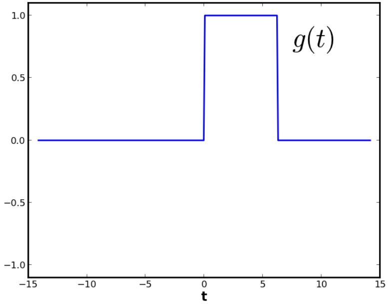
$$C(\tau) = \int_{-\infty}^{\infty} g(\tau - t) f(t)\, dt$$

$$\text{where } f(t) = \sin t \text{ for } 0 < t \leq 2\pi$$

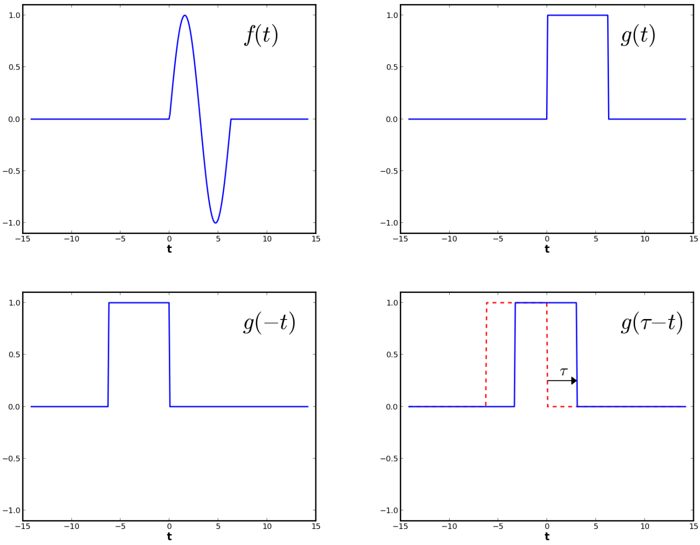
$$g(t) = 1 \text{ for } 0 < t \leq 2\pi$$

and both zero elsewhere.

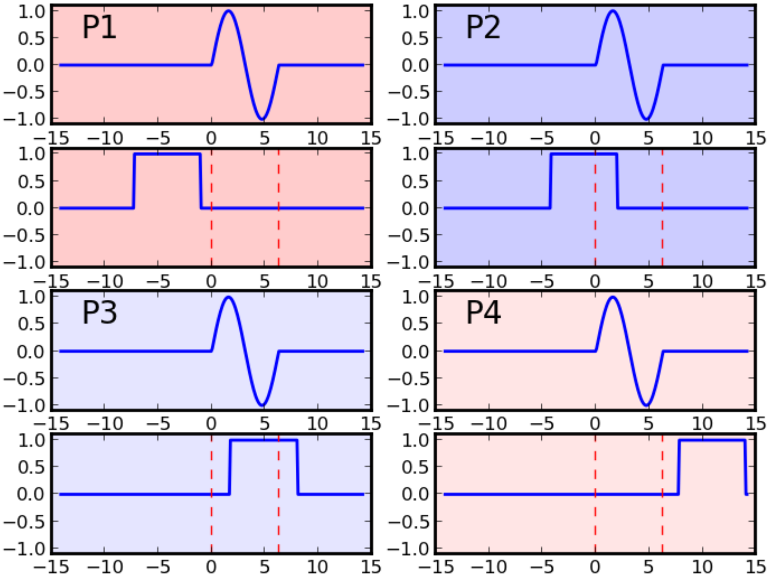




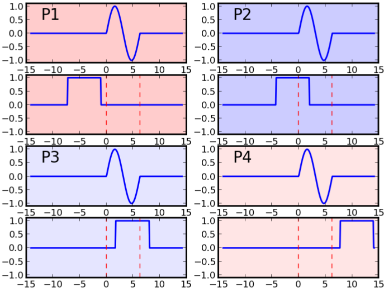
Convolution Example 1



Convolution Example 1 - 4 parts



Convolution Example 1 - 4 parts



P1: No overlap, so contribute zero to convolution

P2: Overlap for $0 < \tau < 2\pi$. Integration limits 0 to τ.

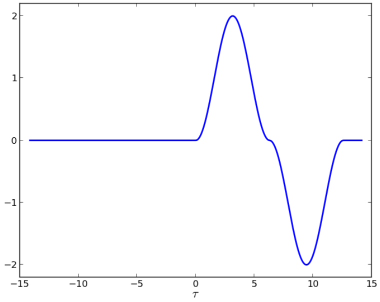
P3: Overlap for $2\pi < \tau < 4\pi$. Integration limits $\tau - 2\pi$ to 2π.

P4: No overlap, $\tau > 4\pi$.

Convolution Example 1 - 4 parts

$$
\begin{aligned}
P2:\ C(\tau) &= \int_0^{\tau} g(\tau - t) f(t)\, dt \\
&= [-\cos t]_0^{\tau} \\
&= -[\cos\tau - \cos 0] = 1 - \cos\tau \\
P3:\ C(\tau) &= \int_{\tau-2\pi}^{2\pi} \sin t\, dt \\
&= [-\cos t]_{\tau-2\pi}^{2\pi} \\
&= -[\cos 2\pi - \cos\tau - 2\pi] = \cos\tau - 1
\end{aligned}
$$

Convolution Example 1 - Result



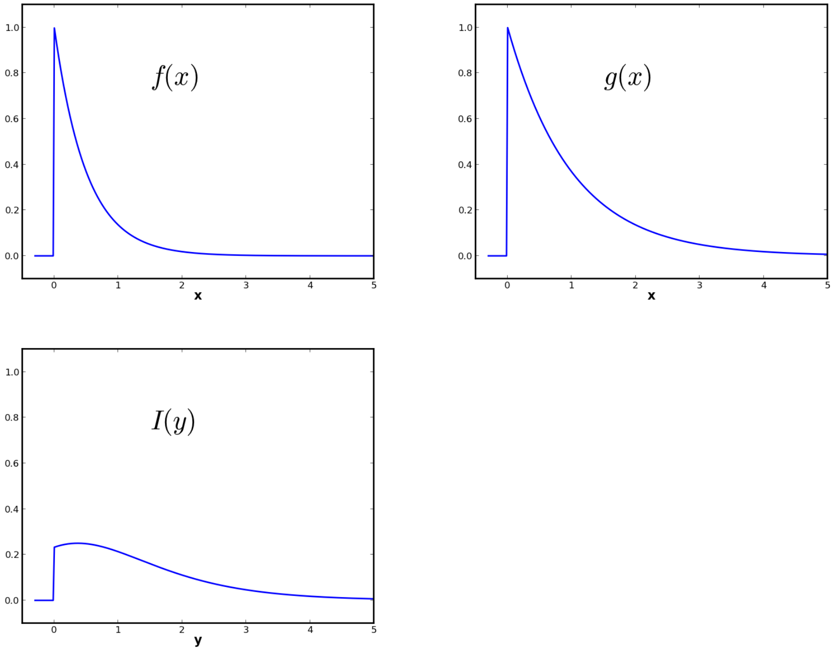
$$C(\tau) = \begin{cases} 0, & \tau < 0 \\ 1 - \cos\tau, & 0 < \tau < 2\pi \\ \cos\tau - 1, & 2\pi < \tau < 4\pi \\ 0, & \tau > 4\pi \end{cases}$$

Convolution Example 2

Convolution of two truncated exponentials

$$
\begin{aligned}
f(x) &= e^{-\alpha x},\ x > 0 \\
g(x) &= e^{-\beta x},\ x > 0
\end{aligned}
$$

$$
\begin{aligned}
I(y) &= \int_{-\infty}^{\infty} f(x) g(y-x)\, dx \\
&= \int_0^y e^{-\alpha x} e^{-\beta(y-x)}\, dx \\
&= e^{-\beta y} \int_0^y e^{-x(\alpha-\beta)}\, dx \\
&= e^{-\beta y} \frac{-1}{\alpha-\beta} \left[e^{-x(\alpha-\beta)} \right]_0^y \\
&= \frac{e^{-\beta y}}{\alpha-\beta} \left[1 - e^{-y(\alpha-\beta)} \right] \\
&= \frac{e^{-\beta y} - e^{-\alpha y}}{\alpha-\beta}
\end{aligned}
$$



$f(x)$
$g(x)$
$I(y)$
x
x
y
0.0
0.2
0.4
0.6
0.8
1.0
0
1
2
3
4
5

FT of a convolution

$$
\begin{aligned}
G(k) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} dx\, e^{-ikx} \int_{-\infty}^{\infty} du\, f_1(x-u) f_2(u) \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} dx\, e^{-ikx} f_1(x-u) f_2(u)
\end{aligned}
$$

Change variable from x to w

$$
w = x - u \qquad dw = dx
$$

$$
\begin{aligned}
G(k) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} dw\, e^{-ik(w+u)} f_1(w) f_2(u) \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} dw\, e^{-ikw} e^{-iku} f_1(w) f_2(u) \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} dw\, e^{-ikw} f_1(w) \int_{-\infty}^{\infty} du\, e^{-iku} f_2(u) \\
&= g_1(k)\, 2\pi g_2(k)
\end{aligned}
$$

Convolution theorem

Convolution Theorem

The Fourier transform of a convolution of two functions is (2π times) the product of their Fourier transforms

$$G(k) \;=\; 2\pi g_1(k) g_2(k)$$