Corrections and Clarifications

In the previous lecture while working on the board and substituting the exponential trail solution into the equation for simple harmonic motion, a factor of i was missed.

The auxillary equation is $\lambda^2 = -k^2$ so $\lambda = \pm ik$.

Outline

- Fourier Series
- General Fourier Series
- Orthogonality
- Complex Exponential Series

Fourier Sine Series

$$\begin{aligned} f(x) &= b_1 \sin\frac{\pi x}{L} + b_2 \sin\frac{2\pi x}{L} + \dots \\ &= \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} \end{aligned}$$

Coefficients b_n calculated through a simple integral.

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}d\,x$$

Works because sin has a special property.

$$\int_0^L \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}\,dx = \begin{cases} 0 & n \neq m \\ \frac{L}{2} & n = m \end{cases}$$

Tool for finding the coefficients in our series sum.

Orthogonality

Definition: Two real functions $v(x)$ and $u(x)$ are orthogonal on the interval $a \leq x \leq b$ if

$$\int_a^b u(x)v(x)\,dx \quad = \quad 0$$

General Fourier Series

More general series, defined on $-L \leq x \leq L$

$$
\begin{aligned}
f(x) &= a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L} \\
&= a_0 + a_1 \cos\frac{\pi x}{L} + b_1 \sin\frac{\pi x}{L} + a_2 \cos\frac{2\pi x}{L} + \dots
\end{aligned}
$$

$\sin\frac{n\pi x}{L}$ and $\cos\frac{n\pi x}{L}$ form a set of orthogonal basis functions on $-L$ to $+L$.

Orthogonality

$$
\begin{aligned}
\int_{-L}^{L} \sin(n\pi x/L)\sin(m\pi x/L)\,dx &= 0 \\
\int_{-L}^{L} \cos(n\pi x/L)\cos(m\pi x/L)\,dx &= 0 \\
\int_{-L}^{L} \sin(n\pi x/L)\cos(m\pi x/L)\,dx &= 0
\end{aligned}
$$

for $n \neq m$ and

$$
\begin{aligned}
\int_{-L}^{L} \sin^2(n\pi x/L)\,dx &= L \\
\int_{-L}^{L} \cos^2(n\pi x/L)\,dx &= L
\end{aligned}
$$

Coefficients

$$
\begin{aligned}
b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx \\
a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx \\
a_0 &= \frac{1}{2L}\int_{-L}^{L} f(x)\,dx
\end{aligned}
$$

$n = 1, 2, 3, ...$

Example

$$f(x) = \begin{cases} 0 & -L \leq x < 0 \\ L - x & 0 < x \leq L \end{cases}$$

Dirichlet's conditions

1807: Fourier publishes series expansion.
1827: Dirichlet's conditions that Fourier series expansion converges to function $f(x)$:

- $f(x)$ must be single valued
- $f(x)$ must have a finite number of finite discontinuities
- $\int_{-L}^{L} |f(x)| \, dx$ must be finite

1966: Proof by Lennart Carleson (2006 Abel Prize) that the set $\sin(n\pi x/L)$ and $\cos(n\pi x/L)$ is complete.

Complex exponential series

Compact notation for combinations of sin and cos

Properties

Multiplication	$e^A e^B = e^{(A+B)}$ $e^{i\theta} e^{i\phi} = e^{i(\theta+\phi)}$
Complex conjugation	$(i \to -i)$, $z = x + iy \to z^* = x - iy$ $(e^{i\theta})^* = e^{-i\theta}$
Modulus squared	$\|z\|^2 = zz^* = x^2 + y^2$ $\|e^{i\theta}\|^2 = e^{i\theta} e^{-i\theta} = e^0 = 1$

Complex exponential series

$$
\begin{aligned}
f(x) &= \sum_{m=-\infty}^{m=\infty} c_m e^{im\pi x/L} \\
&= \ldots + c_{-1} e^{-i\pi x/L} + c_0 + c_1 e^{i\pi x/L} + \ldots
\end{aligned}
$$

Two complex functions $u(x)$ and $v(x)$ are orthogonal on $a \leq x \leq b$ if

$$\int_a^b u^*(x) v(x)\, dx = 0$$

Complex exponentials orthogonal on $-L$ to L:

$$
\begin{aligned}
\textstyle\int_{-L}^{L} \left(e^{in\pi x/L}\right)^* e^{im\pi x/L}\, dx &= 0 && \text{if } m \neq n \\
&= 2L && \text{if } m = n \\
\rightarrow c_n &= \tfrac{1}{2L} \textstyle\int_{-L}^{L} e^{-in\pi x/L} f(x)\, dx
\end{aligned}
$$

Sets of orthogonal functions

Definition: A set of complex functions $u_n(x)$ is orthogonal on $a \leq x \leq b$ if

$$\int_a^b u_m^*(x) u_n(x)\, dx = 0$$

for $m \neq n$. For $m = n$,

$$\int_a^b u_n^*(x) u_n(x)\, dx = \int_a^b |u_n(x)|^2\, dx = I_n > 0$$

(If $I_n = 1$ for all n, functions are ortho-normal.)For a complete set of such functions, can use then as basis functions and write

$$f(x) = \sum_m c_m u_m(x)$$

for any $f(x)$ satisfying Dirichlet's conditions. (More general Fourier series.)c_m are given by

$$c_m = \frac{1}{I_m} \int_a^b u_m^*(x) f(x)\, dx$$