PHYS 20171 MATHEMATICS OF WAVES AND FIELDS

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Some example partial differential equations (PDEs).

- $$\nabla^2 \phi(\mathbf{r}) = \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2}$$

  The wave equation with a constant speed c.
  $\phi(\mathbf{r}, t)$ is the displacement of a vibrating string or component of the **E** field. In 1-D

  $$\frac{\partial^2 \phi(x, t)}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2}$$

- $$\nabla^2 \phi(\mathbf{r}) = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$$

  Laplace's equation
  $\phi(\mathbf{r})$ - electrostatic potential in a region of space without electric charge, or distribution of temperature inside some body.

- $$-\frac{\hbar^2}{2m}\nabla^2\psi(\mathbf{r},t) + V(\mathbf{r})\psi(\mathbf{r},t) = \mathrm{i}\hbar\frac{\partial\psi}{\partial t}$$

  The Schrödinger equation.
  This is the central equation of quantum mechanics.

- $$\nabla^2\phi(\mathbf{r},t) = \frac{1}{D}\frac{\partial\phi}{\partial t}$$

  The heat-flow (or diffusion) equation.

$(\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2})$

Course Outline

1. **ODEs** ($\sim$ 1 lecture)
2. **Fourier series and transforms** ($\sim$ 4 lectures)
3. **Wave problems in one dimension** ($\sim$ 2 lectures)
4. **Other PDE's** ($\sim$ 2 lectures)
5. **Integral transforms** ($\sim$ 2 lectures)
6. **Series solution of ODE's** ($\sim$ 4 lectures)
7. **Problems in two and three dimensions** ($\sim$ 6 lectures
8. **Dirac Notation** ($\sim$ 2 lectures)

Recommended books

- **M. L. Boas,** *Mathematical methods in the physical sciences*, **3rd edn., (Wiley, 2006).**
- **K. F. Riley, M. P. Hobson and S. J. Bence,** *Mathematical Methods for Physics and Engineering* **(Cambridge, 1997)**

Also useful book: G. Stephenson, *Partial differential equations for scientists and engineers* (Imperial College, 1996).

Outline

- ODEs
    - Principle of Superposition
    - Two second order ODEs
    - Complex exponentials

1: Ordinary Differential Equations

Some definitions:

Consider an ODE which has solution ϕ.

- **Order of ODE:** Highest derivative involved in equation.
- **Linear:** Equation does not involved powers of solution ϕ or its derivatives
- **Homogenous:** Every term involves ϕ or its derivatives.

(Similarly for PDEs.)

First order, linear ODEs

$$\frac{df}{dx} = -\alpha f$$

f is a function of one variable, x.
α: constant
Physics: Radioactive decay
Maths: rate of change proportional to function itself.
1 - Trial solution. Guess $f(x) = Ae^{\lambda x}$.
2 - Separate variables and integrate.

Second-order, linear ODEs

$$a\frac{d^2f}{dx^2} + b\frac{df}{dx} + cf \quad = \quad 0$$

Again f is only dependent on one variable, x, so $f(x)$
Constant coefficients (a, b, c)
Homogeneous (RHS=0) - no driving force
Try exponential solution $f = Ae^{\lambda x}$

Superposition

Principle of Superposition: A linear superposition (sum) of solutions to a linear equation is also a solution to the equation.

If $f_1(x)$ and $f_2(x)$ are solutions of a particular ODE, then $G(x) = Af_1(x) + Bf_2(x)$ where A and B are constants, is also a solution.

A Simple (But Important) Second Order ODE

$$\frac{d^2 f}{dx^2} = \alpha^2 f$$

$$\begin{aligned} \text{Auxillary equation} \qquad & \lambda^2 = \alpha^2 \to \lambda = \pm\alpha \\ \text{General solution} \qquad & f(x) = Ce^{\alpha x} + De^{-\alpha x} \\ \text{or} \qquad & f(x) = C' \cosh \alpha x + D' \sinh \alpha x \end{aligned}$$

Simple Harmonic Motion

$$\frac{d^2 f}{dx^2} = -k^2 f$$

sin and cos solutions.
Try an exponential solution: $f(x) = e^{\lambda x}$

Complex exponentials

Terms like

$$e^{ikx}$$

Easy to differentiate. Essential for quantum mechanics.

$$\begin{aligned} \frac{d}{dx} e^{ikx} &= ike^{ikx} \\ \frac{d^2}{dx^2} e^{ikx} &= (ik)^2 e^{ikx} = -k^2 e^{ikx} \end{aligned}$$

Equation of SHM with $f(x) = e^{ikx}$ as a solution.

Complex Solutions SHM

Showed that

$$f = Ce^{\lambda x} \quad \text{where} \quad \lambda = \pm ik$$

$$\text{is a solution of} \qquad \frac{d^2 f}{dx^2} = -k^2 f$$

$$\text{So general solution is} \qquad f(x) = Ce^{ikx} + De^{-ikx}$$