Waves on a sphere

Spherical polar coordinates

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}$$

Wave equation

$$\nabla^2 f = \frac{1}{c^2}\frac{\partial^2 f}{\partial t^2}$$

Waves on surface of a sphere - earthquake ground waves.
Independent of $r : f(\theta,\phi,t) = Y(\theta,\phi)T(t)$.
Boundary conditions:
- periodic in ϕ: $f(\theta,\phi,t) = f(\theta,\phi+2\pi,t)$
- regular everywhere, in particular at poles: $\theta = 0,\pi$ (where $\sin\theta = 0$).

Separate variables ($r = a$)

$$\frac{1}{Ya^2 \sin\theta}\left(\sin\theta \frac{\partial Y}{\partial\theta}\right) + \frac{1}{Ya^2 \sin\theta}\frac{\partial^2 Y}{\partial\phi^2} = \frac{1}{c^2 T}\frac{d^2 T}{dt^2}$$
$$= -\frac{l(l+1)}{a^2}$$

$T(t)$ equation

$$\frac{1}{c^2 T}\frac{d^2 T}{dt^2} = -\frac{l(l+1)}{a^2}$$
$$\frac{d^2 T}{dt^2} = -\omega_l^2 T$$
$$\omega_l^2 = \frac{c^2}{a^2} l(l+1)$$
$$\text{Solution}: T(t) = A\cos\omega_l t + B\sin\omega_l t$$

$Y(\theta, \phi)$ (angular) equation

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right) + \frac{1}{\sin\theta}\frac{\partial^2 Y}{\partial\phi^2} = -l(l+1)Y$$

(Equivalent to

$$\hat{L}Y(\theta, \phi) = l(l+1)\hbar^2 Y(\theta, \phi)$$

in QM)

Separate again $Y(\theta, \phi) = \Theta(\theta)\Phi(\phi)$
Sub., divide by $\Theta(\theta)\Phi(\phi)$ and multiply by $\sin^2\theta$

$$\begin{aligned}
\frac{\sin\theta}{\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + l(l+1)\sin^2\theta + \frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} &= 0 \\
\frac{\sin\theta}{\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + l(l+1)\sin^2\theta &= -\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} \\
&= m^2
\end{aligned}$$

Variables now separated and the terms must equal a constant.

$$\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = -m^2$$

Φ equation

$$\frac{d^2\Phi}{d\phi^2} = -m^2\Phi$$

subject to $\Phi(\phi) = \Phi(\phi + 2\pi)$.
Solutions:

$$\begin{aligned} \Phi(\phi) &= \cos m\phi \\ \text{and } \Phi(\phi) &= \sin m\phi \\ \text{alternatively } \Phi(\phi) &= \exp(\pm im\phi) \end{aligned}$$

and m is integer.

Θ equation

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)+\left(l(l+1)-\frac{m^2}{\sin^2\theta}\right)\Theta \quad = \quad 0$$

Change variables to $x = \cos\theta$ so

$$dx = -\sin\theta\, d\theta \qquad \frac{d}{dx} = -\frac{1}{\sin\theta}\frac{d}{d\theta}$$

$$\sin^2\theta = 1 - x^2$$

Giving

$$\frac{d}{dx}\left(\left(1-x^2\right)\frac{d\Theta}{dx}\right)+\left(l(l+1)-\frac{m^2}{1-x^2}\right)\Theta \quad = \quad 0$$

$$\frac{d}{dx}\left(\left(1-x^2\right)\frac{d\Theta}{dx}\right)+\left(l(l+1)-\frac{m^2}{1-x^2}\right)\Theta \;=\; 0$$

is the associated Legendre equation. (Reduces to the Legendre equation for $m = 0$.)
The solutions are $P_l^m(x)$ - the associated Legendre functions.
These are regular at $x = \pm 1$ provided the series terminates
$\rightarrow l$ is an integer and $|m| <\leq l$.
So

$$\Theta(\theta) \;=\; P_l^m(\cos\theta))$$

The full normal modes are then

$$f(\theta, \phi, t) = P_l^m(\cos\theta)\,(A\cos\omega_l t + B\sin\omega_l t)\exp(\pm im\phi)$$

Symmetric modes; $m = 0$

$$f(\theta, \phi, t) = P_l(\cos\theta)\,(A\cos\omega_l t + B\sin\omega_l t)$$

Orthogonality condition
Need to evaluate by integrating over the surface element on the sphere $a^2 \sin\theta\, d\theta\, d\phi$:

$$\begin{aligned}
\int_0^\pi P_l(\cos\theta) P_k(\cos\theta) dA &= \int_0^\pi P_l(\cos\theta) P_k(\cos\theta) \sin\theta\, d\theta \\
\rightarrow \int_{-1}^{+1} P_l(x) P_k(x)\, dx &= 0 \text{ if } l \neq k \\
&= \frac{2}{2l+1} \text{ if } l = k
\end{aligned}$$

General symmetric wave

$$f(\theta, t) = \sum_{l=0}^{\infty} P_l(x)\,(A_l \cos \omega_l t + B_l \sin \omega_l t)$$

where $x = \cos \theta$.
Want solution given initial shape and velocity of wave, e.g.

$$f(\theta, 0) = G(x) \qquad \left.\frac{\partial f}{\partial t}\right|_{t=0} = 0$$

Use Fourier methods to find A_k and B_k.
At $t = 0$,

$$f(\theta, 0) = \sum_{l=0}^{\infty} A_l P_l(x) = G(x)$$

Multiply by $P_k(x)$ and integrate

$$\sum_l A_l \int_{-1}^{1} P_k(x) P_l(x)\, dx = A_k \frac{2}{2k+1} \quad = \quad \int_{-1}^{1} P_k(x) G(x)\, dx$$

Initial velocity zero so all $B_k = 0$.

Model an earthquake

Represent the impact as a Dirac δ-function, so

$$G(x) = H\delta(x-1)$$

($x = \cos\theta = 1 \to \theta = 0$ - put z-axis through impact site.)
Amplitude of modes is

$$\begin{aligned} A_l &= \frac{2l+1}{2} H \int_{-1}^{1} P_l(x)\delta(x-1)\,dx \\ &= \frac{2l+1}{2} H P_l(1) = \frac{2l+1}{2} H \end{aligned}$$

($P_l(1) = 1$).
Resulting earthquake surface wave

$$f(\theta,t) = \frac{H}{2} \sum_{l=0}^{\infty} (2l+1)\, P_l(x) \cos\omega_l t$$

where $\omega_l = c\sqrt{l(l+1)}/a \approx cl/a$ for $l >> 1$.
At opposite side of the world, $\cos\theta = -1, \theta = \pi$, $P_l(-1) = (-1)^l$

At $t = \pi a/c$, then

$$\begin{aligned} f(\pi, \frac{\pi a}{c}) &= \frac{H}{2} \sum_{l=0}^{\infty} (2l+1)\, P_l(-1) \cos\left(\frac{\omega_l \pi a}{c}\right) \\ &= \frac{H}{2} \sum_{l} (2l+1)(-1)^l(-1)^l \end{aligned}$$

- infinite - massive earthquake as the waves from the impact have travelled half way around the world.