Temperature distribution in a hot plate

Calculate the temperature distribution f in a circular hot plate with an insulated edge at radius a .
Work in plane polar coordinates (r, ϕ).
Heat flow equation for temperature $f(r, \phi, t)$

$$\nabla^2 f \quad = \quad \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial r}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \phi^2} = \frac{1}{D}\frac{\partial f}{\partial t}$$

Subject to

$$\left|\frac{\partial f}{\partial r}\right|_{r=a} a \quad = \quad 0$$

Separate variables: Try $f(r, \phi, t) = R(r)\Phi(\phi)T(t)$. Substitute and divide by $R\Phi T$.

$$\frac{1}{Rr}\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \frac{1}{\Phi r^2}\frac{d^2\Phi}{d\phi^2} \quad = \quad \frac{1}{DT}\frac{dT}{dt} = -k^2$$

Space part: Multiply by r^2 and rearrange

$$\frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right) + k^2 r^2 + \frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} \quad = \quad 0$$

Equation for Φ

$$\frac{d^2\Phi}{d\phi^2} \quad = \quad -m^2\Phi$$

with general solution $\Phi = A\cos m\phi + B\sin m\phi$

But solution must be periodic on $\phi \to \phi + 2\pi$, so m must be integer.
Radial equation

$$r^2 \frac{d^2 R}{dr^2} + r \frac{dR}{dr} + \left(k^2 r^2 - m^2\right) R = 0$$

- Bessel's equation with solutions $J_m(kr)$ and $N_m(kr)$.
Boundary conditions: i) $R(r)$ must be finite at $r = 0$ - regular Bessel functions, $R(r) = J_m(kr)$
and ii) at edge of plate $R'(a) = 0$, so $J'(ka) = 0$
so ka must be a stationary point of $J_m(x)$ and the allowed values of k are quantised.
Finally, T:

$$\frac{dT}{dt} = -\gamma T$$

where $\gamma = Dk^2$ and so $T(t) = Ae^{-\gamma t}$.

Full separable solutions are:

$$f(r,\phi,t) \quad = \quad J_m(kr)\,(A_m \cos m\phi + B_m \sin m\phi)\, e^{-\gamma t}$$

General solution is a linear superposition of these.