SHM

$$\begin{aligned} \frac{d^2y}{dx^2} &= -y \\ \frac{d^2y}{dx^2} + y = y'' + y &= 0 \end{aligned}$$

Solve the hard way. Assume a series solution:

$$\begin{aligned} y(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \ldots \\ y''(x) &= 2a_2 + 6a_3 x + 12 a_4 x^2 + \ldots \end{aligned}$$

Substitute into equation and collect terms

$$y'' + y = 2a_2 + a_0 + (6a_3 + a_1)\, x + (12a_4 + a_2)\, x^2 + \ldots = 0$$

Each term is independent, so coefficients must vanish

$$\begin{aligned}
x^0: &\quad 2a_2 + a_0 = 0 &\rightarrow&\quad a_2 = -\tfrac{1}{2}a_0 \\
x^1: &\quad 6a_3 + a_1 = 0 &\rightarrow&\quad a_3 = -\tfrac{1}{6}a_1 \\
x^2: &\quad 12a_4 + a_2 = 0 &\rightarrow&\quad a_4 = -\tfrac{1}{12}a_2 \\
&&&\qquad = \tfrac{1}{24}a_0
\end{aligned}$$

General n:

$$y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$$

Define new dummy index $m = n - 2$

$$y'' = \sum_{m=0}^{\infty} (m+2)(m+1)a_{m+2}x^m$$

Rename m as n. (It doesn't matter what it's called.)

$$y'' = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n$$

So the ODE is now

$$\sum_{n=0}^{\infty} \left((n+2)(n+1)\,a_{n+2} + a_n\right) x^n = 0$$

Coefficients must vanish, so

$$\begin{aligned}
(n+2)\,(n+1)\,a_{n+2} + a_n &= 0 \\
a_{n+2} &= -\frac{1}{(n+2)(n+1)}a_n \\
&= (-1)^2\frac{1}{(n+2)(n+1)n(n-1)}a_{n-2} \\
a_n &= \begin{cases} (-1)^{n/2}\,\frac{1}{n!}a_0 & \text{even } n \\ (-1)^{(n-1)/2}\,\frac{1}{n!}a_1 & \text{odd } n \end{cases}
\end{aligned}$$

Hence the solution is

$$\begin{aligned} y(x) &= a_0 \left(1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \dots\right) \\ &\quad + a_1 \left(1 - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \dots\right) \\ &= a_0 \cos x + a_1 \sin x \end{aligned}$$