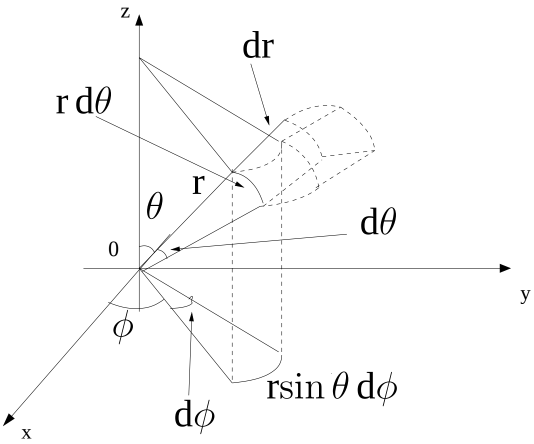
PC10372, Mathematics 2
Workshop Sheet 9 Solutions

1)



$d\tau = r\,d\theta\,r\sin\theta\,d\phi\,dr = r^2\sin\theta\,dr\,d\theta\,d\phi$

Areas of faces:
Lower: $r\sin\theta\,d\phi\,r\,d\theta = r^2\sin\theta\,d\theta\,d\phi$
Upper: $(r+dr)\sin\theta\,d\phi\,(r+dr)d\theta = (r+dr)^2\sin\theta\,d\theta\,d\phi$
Back: $r\sin\theta\,d\phi\,dr$
Front: $r\sin\theta\,d\phi\,dr$
Left: $r\,dr\,d\theta$
Right: $r\,dr\,d\theta$

2) So the vector areas are:
Lower: $\underline{dS}_{\text{lower}} = -r^2\sin\theta\,d\theta\,d\phi\,\underline{\hat{\mathbf{r}}}$
Upper: $\underline{dS}_{\text{upper}} = (r+dr)^2\sin\theta\,d\theta\,d\phi\,\underline{\hat{\mathbf{r}}}$
Back: $\underline{dS}_{\text{left}} = -r\sin\theta\,dphi\,dr\,\underline{\hat{\boldsymbol{\theta}}}$
Front: $\underline{dS}_{\text{right}} = r\sin\theta\,d\phi\,dr\,\underline{\hat{\boldsymbol{\theta}}}$
Left: $\underline{dS}_{\text{front}} = -r\,d\theta\,dr\,\underline{\hat{\boldsymbol{\phi}}}$
Right: $\underline{dS}_{\text{back}} = r\,d\theta\,dr\,\underline{\hat{\boldsymbol{\phi}}}$

3) Flux out through upper surface $= F_{\text{upper}}$

$$F_{\text{upper}} = \underline{\mathbf{v}} \cdot \underline{\mathbf{d}S} = v_r(r+dr,\theta,\phi)(r+dr)^2\sin\theta\,d\theta\,d\phi$$

$$= \left(r^2 v_r(r,\theta,\phi) + \frac{\partial}{\partial r}\left(r^2 v_r(r,\theta,\phi)\right) dr\right) \sin\theta \, d\theta \, d\phi$$

where the second step uses the Taylor series expansion to expand the $v_r(r+dr),\theta,\phi)(r+dr)^2)$ term. Now for the lower surface

$$F_{\text{lower}} = \underline{\mathbf{v}} \cdot \underline{\mathbf{d}S} = v_r(r,\theta,\phi)(-r^2) \sin\theta \, d\theta \, d\phi$$

Adding the flux through these two faces gives

$$F_{\text{lower}} + F_{\text{upper}} = \frac{\partial}{\partial r}\left(r^2 v_r(r,\theta,\phi)\right) \sin\theta \, dr \, d\theta \, d\phi$$

Now for the left and right faces:

$$\begin{aligned}
F_{\text{left}} &= \underline{\mathbf{v}} \cdot \underline{\mathbf{d}S} = -v_\phi(r,\theta,\phi) \, r \, dr \, d\theta \\
F_{\text{right}} &= \underline{\mathbf{v}} \cdot \underline{\mathbf{d}S} = v_\phi(r,\theta,\phi+d\phi) \, r \, dr \, d\theta \\
&= \left(v_\phi(r,\theta,\phi) + \frac{\partial v_\phi}{\partial \phi} d\phi\right) r \, dr \, d\theta \\
F_{\text{left}} + F_{\text{back}} &= r \frac{\partial v_\phi}{\partial \phi} dr \, d\theta \, d\phi
\end{aligned}$$

Now for the front and back faces:

$$\begin{aligned}
F_{\text{front}} &= v_\theta(r,\theta+d\theta,\phi) r \sin(\theta+d\theta) \, dr \, d\phi \\
&= \left(v_\theta(r,\theta,\phi)\sin\theta + \frac{\partial}{\partial \theta}\left(\sin\theta \, v_\theta\right) d\theta\right) r \, dr \, d\phi \\
F_{\text{back}} &= -v_\theta(r,\theta,\phi) r \sin(\theta) \, dr \, d\phi \\
F_{\text{front}} + F_{\text{back}} &= r \frac{\partial}{\partial \theta}\left(v_\theta \sin\theta\right) dr \, d\theta \, d\phi
\end{aligned}$$

Summing the results for these three pairs of faces and dividing by the element volume gives us the result we are looking for:

$$\nabla \cdot \underline{\mathbf{v}} = \frac{1}{r^2 \sin\theta}\left(\sin\theta \frac{\partial (r^2 v_r)}{\partial r} + r \frac{\partial (\sin\theta \, v_\theta)}{\partial \theta} + r \frac{\partial v_\phi}{\partial \phi}\right)$$