PC10372, Mathematics 2
Workshop Sheet 9

This multi-part question aims to derive the general expression for the divergence in spherical polar coordinates. By the end of this workshop you should have shown that

$$\nabla \cdot \underline{\mathbf{v}} \;\; = \;\; \frac{1}{r^2}\frac{\partial (r^2 V_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta\, v_\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}$$

where $\underline{\mathbf{v}}(r,\theta,\phi) = v_r(r,\theta,\phi)\underline{\hat{\mathbf{r}}} + v_\theta(r,\theta,\phi)\underline{\hat{\boldsymbol{\theta}}} + v_\phi(r,\theta,\phi)\underline{\hat{\boldsymbol{\phi}}}$.

1) Start by sketching a diagram which shows clearly an elemental volume in spherical polar coordinates. Place one corner (vertex) of the element at point (r,θ,ϕ) and the opposite corner (vertex) at $(r+dr,\theta+d\theta,\phi+d\phi)$. A good diagram is very important!

By writing down the product of the lengths of the three edges of the volume element show that the element has volume

$$d\tau = r^2\sin\theta\, dr\, d\theta\, d\phi$$

What are the areas of the six faces of the element?

2) Next, we are going to compute the divergence by calculating the righthand side (rhs) of the formula

$$(\nabla\cdot\underline{\mathbf{v}})\; d\tau \;\; = \;\; \int_S \underline{\mathbf{v}}\cdot\underline{\mathbf{d}A}$$

where the flux integral on the rhs is over the surface S which bounds the infinitesimal volume $d\tau$. Use your result from part 1) above to write down the infinitesimal areas $\underline{\mathbf{d}S}$ associated with each of the siz faces. (Assume that the unit normals are parallel to $\pm\underline{\hat{\mathbf{r}}}$, $\pm\underline{\hat{\boldsymbol{\theta}}}$ and $\pm\underline{\hat{\boldsymbol{\phi}}}$).

3) Now we have to compute the flux out of each of the six faces. Pick any one of the faces and write down $\underline{\mathbf{v}}\cdot\underline{\mathbf{d}S}$ for it. You can now write down the flux out of this face. Now consider the opposite face. What is the flux out of it? One of these fluxes should be more complicated that the other but can be simplified since the box is infinitesimalm so that we can use the Taylor expansion

$$f(x+\delta x)g(x+\delta x) \;\; \approx \;\; f(x)g(x) + \frac{d(f(x)g(x))}{dx}\delta x$$

to write the more commplicated flux up to and including terms which are cubic in the infinitesimal quantities. Add the two fluxes together and repeat the whole procedure for the other two pairs of faces.

4) Once you have the total flux out of the element you can go ahead and divide by the volume of the box, i.e. $d\tau$. If you have done everything correctly you should get the reslut which is given at the top of this page.