PC10372, Mathematics 2
Workshop Sheet 5 Solutions

1) $\frac{\partial^2 f}{\partial x^2} = 0$, $\frac{\partial^2 f}{\partial y^2} = 2xz^3$ and $\frac{\partial^2 f}{\partial z^2} = 6xy^2 z$ so

$$\nabla^2 f = 2xz^3 + 6xy^2 z$$

2) $\frac{\partial^2 \underline{\mathbf{v}}}{\partial x^2} = 3x\underline{\mathbf{k}}$, $\frac{\partial^2 \underline{\mathbf{v}}}{\partial y^2} = 0$ and $\frac{\partial^2 \underline{\mathbf{v}}}{\partial z^2} = 0$ so

$$\nabla^2 \underline{\mathbf{v}} = 3x\underline{\mathbf{k}}$$

3)

$$\begin{aligned}
\underline{\mathbf{r}} \times \mathbf{s} &= \begin{vmatrix} \underline{\mathbf{i}} & \underline{\mathbf{j}} & \underline{\mathbf{k}} \\ x & y & z \\ z & x & y \end{vmatrix} = \underline{\mathbf{i}}\left(y^2 - xz\right) + \underline{\mathbf{j}}\left(z^2 - xy\right) + \underline{\mathbf{k}}\left(x^2 - yz\right) \\
\nabla \times \underline{\mathbf{r}} &= \begin{vmatrix} \underline{\mathbf{i}} & \underline{\mathbf{j}} & \underline{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = 0 \\
\nabla \times \mathbf{s} &= \begin{vmatrix} \underline{\mathbf{i}} & \underline{\mathbf{j}} & \underline{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & x & y \end{vmatrix} = \underline{\mathbf{i}} + \underline{\mathbf{j}} + \underline{\mathbf{k}} \\
\nabla \cdot \underline{\mathbf{r}} &= 1 + 1 + 1 = 3 \\
\nabla \cdot \mathbf{s} &= 0 + 0 + 0 = 0 \\
\nabla \cdot (\underline{\mathbf{r}} \times \mathbf{s}) &= -z - x - y \\
(\nabla \times \underline{\mathbf{r}}) \cdot \mathbf{s} &= 0 \\
(\nabla \times \mathbf{s}) \cdot \underline{\mathbf{r}} &= \left(\underline{\mathbf{i}} + \underline{\mathbf{j}} + \underline{\mathbf{k}}\right) \cdot \left(x\underline{\mathbf{i}} + y\underline{\mathbf{j}} + z\underline{\mathbf{k}}\right) = x + y + z
\end{aligned}$$

Substituting demonstrates that $\nabla \cdot (\underline{\mathbf{r}} \times \mathbf{s}) = (\nabla \times \underline{\mathbf{r}}) \cdot \mathbf{s} - (\nabla \times \mathbf{s}) \cdot \underline{\mathbf{r}}$.

$$\begin{aligned}
\nabla \times (\underline{\mathbf{r}} \times \mathbf{s}) &= \begin{vmatrix} \underline{\mathbf{i}} & \underline{\mathbf{j}} & \underline{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 - zx & z^2 - xy & x^2 - yz \end{vmatrix} \\
&= \underline{\mathbf{i}}(-3z) = \underline{\mathbf{j}}(-3x) + \underline{\mathbf{k}}(-3y) = -3\mathbf{s}
\end{aligned}$$

$$
\begin{aligned}
\underline{\mathbf{r}}(\nabla\cdot\mathbf{s}) - \mathbf{s}(\nabla\cdot\underline{\mathbf{r}}) + (\mathbf{s}\cdot\nabla)\underline{\mathbf{r}} - (\underline{\mathbf{r}}\cdot\nabla)\mathbf{s} &= 0 - 3\mathbf{s} \\
&\quad + \left(z\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} + y\frac{\partial}{\partial z}\right)(x\underline{\mathbf{i}} + y\underline{\mathbf{j}} + z\underline{\mathbf{k}}) \\
&\quad - \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right)(z\underline{\mathbf{i}} + x\underline{\mathbf{j}} + y\underline{\mathbf{k}}) \\
&= -3\mathbf{s} + \left(z\underline{\mathbf{i}} + x\underline{\mathbf{j}} + y\underline{\mathbf{k}}\right) - \left(x\underline{\mathbf{j}} + y\underline{\mathbf{k}} + z\underline{\mathbf{i}}\right) \\
&= -3\mathbf{s}
\end{aligned}
$$

showing that $\nabla\times(\underline{\mathbf{r}}\times\mathbf{s}) = \underline{\mathbf{r}}(\nabla\cdot\mathbf{s}) - \mathbf{s}(\nabla\cdot\underline{\mathbf{r}}) + (\mathbf{s}\cdot\nabla)\underline{\mathbf{r}} - (\underline{\mathbf{r}}\cdot\nabla)\mathbf{s}$

4) Cartesian: $f = x$, $\nabla f = \underline{\mathbf{i}}$
Cylindrical polar: $f = r\cos\theta$, $\nabla f = \cos\theta\hat{\underline{\mathbf{r}}} - \sin\theta\hat{\underline{\boldsymbol{\theta}}}$
Spherical polar: $f = r\sin\theta\cos\phi$, $\nabla f = \sin\theta\cos\phi\hat{\underline{\mathbf{r}}} + \cos\theta\cos\phi\hat{\underline{\boldsymbol{\theta}}} - \sin\phi\hat{\underline{\boldsymbol{\phi}}}$

Since the results must be independent of the coordinate system, these results must all the same so equating the Cartesian and spherical polar results gives

$$
\underline{\mathbf{i}} = \sin\theta\cos\theta\hat{\underline{\mathbf{r}}} + \cos\theta\cos\phi\hat{\underline{\boldsymbol{\theta}}} - \sin\phi\hat{\underline{\boldsymbol{\phi}}}
$$

as required.