PC10372, Mathematics 2
Workshop Sheet 5

Recall that the **Laplacian** is defined as

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

1) Calculate $\nabla^2 f$ were $f(x, y, z) = xy^2z^3$.

2) Compute $\nabla^2\underline{\mathbf{v}}$ where $\underline{\mathbf{v}} = 4x\underline{\mathbf{i}} + xyz\underline{\mathbf{j}} + x^3\underline{\mathbf{k}}$

3) Consider the two vector fields

$$\begin{aligned} \underline{\mathbf{r}}(x, y, z) &= x\underline{\mathbf{i}} + y\underline{\mathbf{j}} + z\underline{\mathbf{k}} \\ \mathbf{s}(x, y, z) &= z\underline{\mathbf{i}} + x\underline{\mathbf{j}} + y\underline{\mathbf{k}} \end{aligned}$$

Calculate $\underline{\mathbf{r}} \times \mathbf{s}$, $\nabla \times \underline{\mathbf{r}}$, $\nabla \times \mathbf{s}$, $\nabla \cdot \underline{\mathbf{r}}$, $\nabla \cdot \mathbf{s}$, $\nabla \cdot (\underline{\mathbf{r}} \times \mathbf{s})$ and $\nabla \times (\underline{\mathbf{r}} \times \mathbf{s})$. Use all these results to demonstrate that

$$\nabla \cdot (\underline{\mathbf{r}} \times \mathbf{s}) = (\nabla \times \underline{\mathbf{r}}) \cdot \mathbf{s} - (\nabla \times \mathbf{s}) \cdot \underline{\mathbf{r}}$$

and

$$\nabla \times (\underline{\mathbf{r}} \times \mathbf{s}) = \underline{\mathbf{r}}\,(\nabla \cdot \mathbf{s}) - \mathbf{s}\,(\nabla \cdot \underline{\mathbf{r}}) + (\mathbf{s} \cdot \nabla)\,\underline{\mathbf{r}} - (\underline{\mathbf{r}} \cdot \nabla)\,\mathbf{s}$$

4) In lecture we showed how to compute the gradient of a scalar field in spherical and cylindrical polar coordinates:

$$\begin{aligned} \nabla f &= \frac{\partial f}{\partial r}\underline{\hat{\mathbf{r}}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\underline{\hat{\boldsymbol{\theta}}} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\underline{\hat{\boldsymbol{\phi}}} \\ \nabla f &= \frac{\partial f}{\partial r}\underline{\hat{\mathbf{r}}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\underline{\hat{\boldsymbol{\theta}}} + \frac{\partial f}{\partial z}\underline{\mathbf{k}} \end{aligned}$$

Use these results to determine, in both spherical and cylindrical polar coordinates, $\nabla f(x, y, z)$ where $f(x, y, z) = x$ and hence show that

$$\underline{\mathbf{i}} = \sin\theta\cos\theta\underline{\hat{\mathbf{r}}} + \cos\theta\cos\phi\underline{\hat{\boldsymbol{\theta}}} - \sin\phi\underline{\hat{\boldsymbol{\phi}}}$$