PC10372, Mathematics 2
Workshop Sheet 4 Solutions

1)

$$\nabla \cdot \mathbf{v} \;=\; \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

so

$$\begin{aligned}
i)\ \nabla \cdot \mathbf{v} &= \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(2y) + \frac{\partial}{\partial z}(3) = 1 + 2 + 0 = 3 \\
ii)\ \nabla \cdot \mathbf{v} &= \frac{\partial}{\partial x}(y) + \frac{\partial}{\partial y}(x) + \frac{\partial}{\partial z}(-x\ln y) = 0 + 0 + 0 = 0 \\
iii)\ \nabla \cdot \mathbf{v} &= \frac{\partial}{\partial x}(e^{-x}\sin 2y) + \frac{\partial}{\partial y}(e^{-x}\sin 2y) + \frac{\partial}{\partial z}(e^{-x}\sin 2y) \\
&= -e^{-x}\sin 2y + 2e^{-2}\cos 2y
\end{aligned}$$

2)

$$\nabla \times \mathbf{v} \;=\; \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix}$$

$$\begin{aligned}
i)\ \nabla \times \mathbf{v} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & 2y & 3 \end{vmatrix} \\
&= \mathbf{i}\left(\frac{\partial}{\partial y}(3) - \frac{\partial}{\partial z}(2y)\right) - \mathbf{j}\left(\frac{\partial}{\partial x}(3) - \frac{\partial}{\partial z}(x)\right) + \mathbf{k}\left(\frac{\partial}{\partial x}(2y) - \frac{\partial}{\partial y}(x)\right) = 0
\end{aligned}$$

$$\begin{aligned}
ii)\ \nabla \times \mathbf{v} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & x & -x\ln y \end{vmatrix} \\
&= \mathbf{i}\left(-\frac{x}{y} - 0\right) - \mathbf{j}\left(-\ln y - 0\right) + \mathbf{k}\left(1 - 1\right) = -\frac{x}{y}\mathbf{i} + \ln y\mathbf{j}
\end{aligned}$$

$$\begin{aligned}
iii)\ \nabla \times \mathbf{v} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^{-x}\sin 2y & e^{-x}\sin 2y & e^{-x}\sin 2y \end{vmatrix} \\
&= \mathbf{i}\left(2e^{-x}\cos 2y - 0\right) - \mathbf{j}\left(-e^{-x}\sin 2y - 0\right) + \mathbf{k}\left(-e^{-x}\sin 2y - 2e^{-x}\cos 2y\right) \\
&= 2e^{-x}\cos 2y(\mathbf{i} - \mathbf{k}) + e^{-x}\sin 2y(\mathbf{j} - \mathbf{k})
\end{aligned}$$

$i)\ \nabla \cdot (\nabla \times v) = 0$

$$ii)\ \nabla \cdot (\nabla \times v) \;=\; \frac{\partial}{\partial x}\left(-\frac{x}{y}\right) + \frac{\partial}{\partial y}(\ln y) = -\frac{1}{y} + \frac{1}{y} = 0$$

$$\begin{aligned} iii)\ \nabla \cdot (\nabla \times v) &= \frac{\partial}{\partial x}\left(2e^{-x}\cos 2y\right) + \frac{\partial}{\partial y}\left(e^{-x}\sin 2y\right) + \frac{\partial}{\partial z}\left(-2e^{-x}\cos y - e^{-x}\sin 2y\right) \\ &= -2e^{-x}\cos 2y + 2e^{-x}\cos 2y + 0 = 0 \end{aligned}$$

General proofs:

$$\begin{aligned} \nabla \cdot (\nabla \times \mathbf{v}) &= \nabla \cdot \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix} \\ &= \frac{\partial}{\partial x}\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right) - \frac{\partial}{\partial y}\left(\frac{\partial v_z}{\partial x} - \frac{\partial v_x}{\partial z}\right) + \frac{\partial}{\partial z}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right) \\ &= \frac{\partial^2 v_z}{\partial x \partial y} - \frac{\partial^2 v_y}{\partial x \partial z} - \frac{\partial^2 v_z}{\partial y \partial x} + \frac{\partial^2 v_x}{\partial y \partial z} + \frac{\partial^2 v_y}{\partial z \partial x} - \frac{\partial^2 v_x}{\partial z \partial y} = 0 \end{aligned}$$

because $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$, etc. for any function f.

$$\begin{aligned} \nabla \times (\nabla f) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix} \\ &= \mathbf{i}\left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}\right) - \mathbf{j}\left(\frac{\partial^2 f}{\partial x \partial z} - \frac{\partial^2 f}{\partial z \partial x}\right) + \mathbf{k}\left(\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x}\right) = 0 \end{aligned}$$