PC10372, Mathematics 2
Workshop Sheet 3 Solutions

1)

1. $\nabla f = \mathbf{i} + \mathbf{j} - 2\mathbf{k}$
2. $\nabla f = (10x - z)\mathbf{i} - x\mathbf{k}$
3. $\nabla f = -2e^{-2x}\sin(\pi y)\mathbf{i} + \pi e^{-2x}\cos\pi y\mathbf{j}$

2) The unit vector in the direction is required $\hat{\mathbf{n}} = (\mathbf{i} + \mathbf{j})/\sqrt{2}$ and we need to calculate $\nabla f \cdot \hat{\mathbf{n}}$, evaluated at the origin.

1. $\nabla f \cdot (\frac{\mathbf{i}+\mathbf{j}}{\sqrt{2}}) = \frac{1+1}{\sqrt{2}} = \sqrt{2}$
2. $\nabla f \cdot (\frac{\mathbf{i}+\mathbf{j}}{\sqrt{2}}) = \frac{10x-z}{\sqrt{2}} = 0$ at $x = y = z = 0$.
3. $\nabla f \cdot (\frac{\mathbf{i}+\mathbf{j}}{\sqrt{2}}) = \frac{-2e^{-2x}\sin\pi y}{\sqrt{2}} + \frac{\pi e^{-2x}\cos\pi y}{\sqrt{2}}$. At $x = y = z = 0$ this evaluates to $\pi/\sqrt{2}$.

3) At the origin $x = y = z = 0$

1. $\nabla f = \mathbf{i} + \mathbf{j} - 2\mathbf{k}$
2. $\nabla f = 0$
3. $\nabla f = \pi\mathbf{j}$

The rate of change is greatest in the direction $\mathbf{n} = \nabla f/|\nabla f|$

1. $\mathbf{n} = \frac{1}{\sqrt{6}}(\mathbf{i} + \mathbf{j} - 2\mathbf{k})$, rate of change, $|\nabla f| = \sqrt{6}$
2. $\mathbf{n}$ is undefined, $|\nabla f| = 0$
3. $\mathbf{n} = \mathbf{j}$, $|\nabla f| = \pi$

4) Define the function $g(x, y, z) = x^3y^2z - 1 = 0$. The vector ∇g is normal to the surfaces of constant g.

$$\begin{aligned} \nabla g &= 3x^2y^2z\mathbf{i} + 2x^3yz\mathbf{j} + x^3y^2\mathbf{k} \\ &= 3\mathbf{i} + 2\mathbf{j} + \mathbf{k} \text{ at the point (1,1,1).} \end{aligned}$$

So the vector $\mathbf{n} = 3\mathbf{i}+2\mathbf{j}+\mathbf{k}$ is normal to the surface at (1,1,1) and hence the unit normal is $(3\mathbf{i}+2\mathbf{j}+\mathbf{k})/\sqrt{14}$.

5) $\mathbf{v}(x, y) = e^{-x}\mathbf{i} + \mathbf{j}$. The field lines are given by

$$\begin{aligned} \frac{dy}{dx} &= \frac{v_y(x, y)}{v_x(x, y)} \\ &= \frac{1}{e^{-x}} = e^x \\ \rightarrow y &= e^x + c \end{aligned}$$

were c is a constant. Different field lines correspond to different values of c. So