PC10372, Mathematics 2
Workshop Sheet 1 Solutions

a) The number of deer born depends on the number of deer there are who can have babies. So the more deer, the more births and babies. Hence $dN/dt = kN$ where k is a constant, the birth rate.

   The solution to this equation (as can be seen by direct integration) is $N = N_0 e^{kt}$. In other words the population of deer grows exponentially with time and no matter how small the initial population the planet will eventually be over run with deer.

b) If there is a fixed food supply it could allow the population to grow at a rate limited by the food supply. However if there are already N deer this food supply must also support them and so less is available to increase the number of deer. So the contribution to increase in deer number from the food supply can be written as $M - N$, so $dN/dt \propto (M - N)$ where M is a constant related to the food supply.

c) Notice that dN/dt is proportional to both the terms discussed in a) and b) so combining these we multiply the terms and so get

$$\begin{aligned}
\frac{dN}{dt} &= \alpha(M-N)N \\
\int \frac{dN}{(M-N)N} &= \int \alpha\, dt \\
\text{Convert to partial fractions: } \frac{1}{(M-N)N} &= \frac{A}{N} + \frac{B}{M-N} \\
\rightarrow A(M-N) + BN = 1 & \\
\text{Put } N = 0 \rightarrow A = 1/M \quad & \quad \text{Put } N = M, \rightarrow B = 1/M \\
\therefore \int \frac{dN}{(M-N)N} &= \frac{1}{M} \int \left(\frac{1}{N} + \frac{1}{M-N} \right) dN = \int \alpha\, dt
\end{aligned}$$

   where α is a constant. Integrating we get

$$\begin{aligned}
\frac{1}{M} (\ln N - \ln |M-N|) &= \alpha t + C \\
\frac{N}{M-N} &= K e^{M\alpha t}
\end{aligned}$$

d) Rearranging we get

$$N = \frac{MAe^{M\alpha t}}{1 + Ae^{M\alpha t}}$$

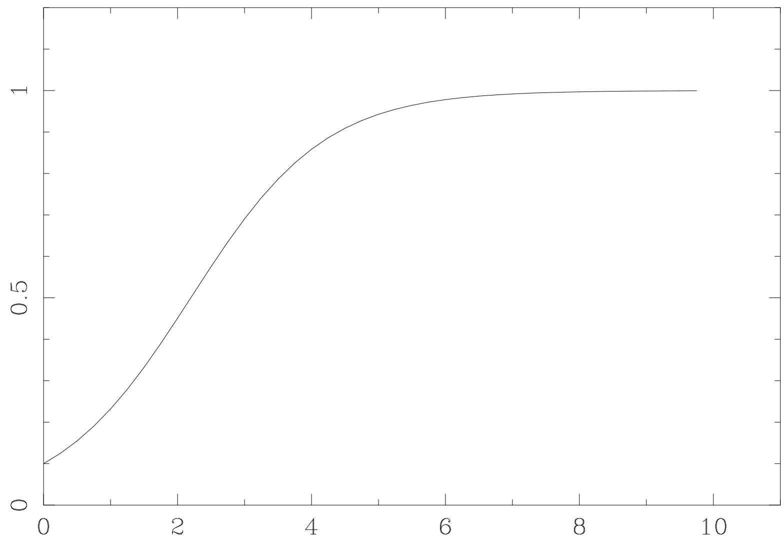
e) At $t = 0$, $N = N_0 = MK/(1+K)$, $K = N_0/(M - N_0)$ which gives

$$N = \frac{M \frac{N_0}{M-N_0} e^{M\alpha t}}{1 + \frac{N_0}{M-N_0} e^{M\alpha t}} = \frac{MN_0 e^{M\alpha t}}{M - N_0 + N_0 e^{M\alpha t}}$$

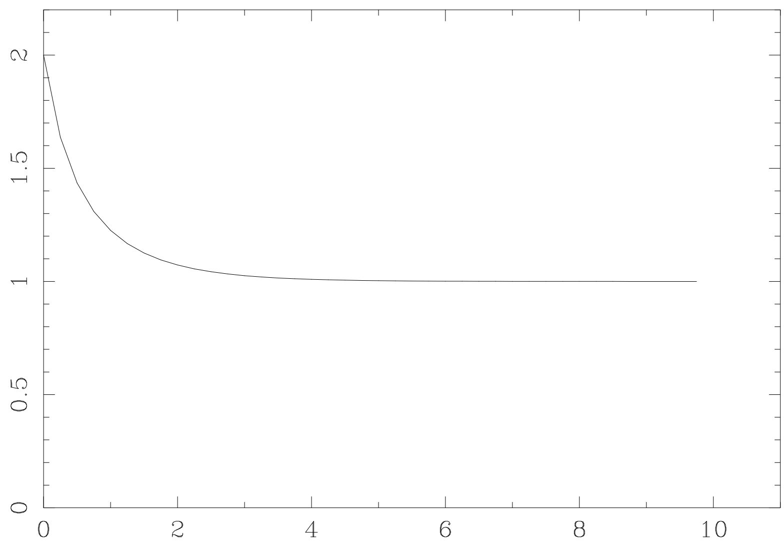
f) Notice that for large t the exponential term in the denominator dominates so

$$N \quad \rightarrow \quad \frac{MN_0 e^{M\alpha t}}{N_0 e^{M\alpha t}} = M = 1$$

i) For small t, $N \sim \frac{N_0 e^t}{1-N_0+N_0} \sim N_0 e^t = 0.1e^t$, so exponential growth if $N_0 < M$. The population starts to grow exponentially then flattens off, tending to M, which is 1 here, at large times.



ii) Dividing the top and bottom by e^t gives $N = 2/(2 - e^{-t})$. This expression is decreasing as t increases, so N falls from 2, tending towards 1 at large times.



g) If there are few deer initially their population grows rapidly until they come into balance with the food supply. If the population starts of large it declines until it's in balance.