PC10372, Mathematics 2
Workshop Sheet 10 Solutions

1)

$$\begin{aligned}
\nabla \cdot \underline{\mathbf{V}} &= 3y - 2x \\
\text{flux} &= \int_0^1 \int_0^1 \int_0^1 (3y - 2x)\, dx\, dy\, dz = 3 \int_0^1 y\, dy \int_0^1 dx \int_0^1 dz - 2 \int_0^1 dx\, x \int_0^1 dy \int_0^1 dz \\
&= 3 \left[\frac{y^2}{2} \right]_0^1 - 2 \left[\frac{x^2}{2} \right]_0^1 = \frac{3}{2} - \frac{2}{2} = \frac{1}{2}
\end{aligned}$$

For the sphere use spherical polar coordinates, so $dV = r^2 \sin\theta\, dr\, d\theta\, d\phi$, and $x = r \sin\theta \cos\phi$, $y = r\sin\theta \sin\phi$

$$\begin{aligned}
\text{flux} &= \int_0^{2\pi} \int_0^{\pi} \int_0^1 (3\sin\phi - 2\cos\phi)\, r \sin\theta\, r^2 \sin\theta\, dr\, d\theta\, d\phi \\
\text{But } \int_0^{2\pi} \sin\phi\, d\phi &= \int_0^{2\pi} \cos\phi\, d\phi = 0 \\
\text{so that flux} &= 0
\end{aligned}$$

2) Q is the charge enclosed, so

$$\begin{aligned}
\frac{Q}{\epsilon_0} &= \int_{\text{hemisphere surface}} \underline{\mathbf{E}} \cdot \underline{\mathbf{d}S} = \int_{\text{hemisphere}} (\nabla \cdot \underline{\mathbf{E}})\, dV \\
\nabla \cdot \underline{\mathbf{E}} &= 2C \\
\text{So } \frac{Q}{\epsilon_0} &= 2C \int dV = 2C \frac{2}{3} \pi R^3 = \frac{4\pi}{3} C R^3
\end{aligned}$$

By direct evaluation:

$$\text{Flux} = C \int_{\text{curved surface}} (x\underline{\mathbf{i}} + y\underline{\mathbf{j}}) \cdot \underline{\hat{\mathbf{r}}} R^2 \sin\theta\, d\theta\, d\phi + C \int_{\text{base}} (x\underline{\mathbf{i}} + y\underline{\mathbf{j}}) \cdot (-\underline{\mathbf{k}}) dx\, dy$$

Note that the second term is zero since $\underline{\mathbf{E}} \cdot \underline{\mathbf{k}} = 0$. Now use $x = R\sin\theta\cos\phi$, $y = R \sin\theta \sin\phi$, $\underline{\mathbf{i}} \cdot \underline{\hat{\mathbf{r}}} = \sin\theta\cos\phi$ and $\underline{\mathbf{j}} \cdot \underline{\hat{\mathbf{r}}} = \sin\theta \sin\phi$, so that

$$\begin{aligned}
\text{flux} &= CR^2R \int_0^{\pi/2} \sin^3\theta\, d\theta \int_0^{2\pi} \left(\sin^2\phi + \cos^2\phi \right) d\phi = CR^3 2\pi \int_0^{\pi/2} \sin\theta (1 - \cos^2\theta)\, d\theta \\
&= 2\pi C R^3 \left[-\cos\theta + \frac{\cos^3\theta}{3} \right]_0^{\pi/2} = \frac{4\pi C R^3}{3}
\end{aligned}$$

3)

$$\int_S \underline{\mathbf{E}} \cdot \underline{\mathbf{d}S} = \frac{Q}{\epsilon_0} = \frac{1}{\epsilon_0} \int_V \rho \, dV$$

since ρ is the charge density, integrating it over the volume gives Q, the total charge. But using the divergence theorem,

$$\int_S \underline{\mathbf{E}} \cdot \underline{\mathbf{d}S} = \int_V \nabla \cdot \underline{\mathbf{E}} \, dV$$

$$\text{and so } \int_V \nabla \cdot \underline{\mathbf{E}} \, dV = \frac{1}{\epsilon_0} \int_V \rho \, dV$$

Since this must be true for all volumes, the integrands must be equal so

$$\nabla \cdot \underline{\mathbf{E}} = \frac{\rho}{\epsilon_0}$$