PC10372, Mathematics 2
Workshop Sheet 10

This week's questions focus on getting practise using the Divergence Theorem:

$$\int_S \underline{\mathbf{u}} \cdot \underline{\mathbf{d}S} \;=\; \int_V (\nabla \cdot \underline{\mathbf{u}})\, dV$$

1) Use the divergence theorem to evaluate the flux of the vector field

$$\underline{\mathbf{V}} \;=\; 3xy\underline{\mathbf{i}} - 2zx\underline{\mathbf{k}}$$

out of the unit cube (i.e. sides the cube have length 1) which has one corner at the origin, its faces lie parallel to the coordinate axes and $x, y, z > 0$ in its interior.

Also evaluate the flux out of a sphere of unit radius which is centred on the origin.

2) An electrostratic field in a vacuum is given by

$$\underline{\mathbf{E}} \;=\; C(x\underline{\mathbf{i}} + y\underline{\mathbf{j}})$$

where C is a constant. Use Gauss's Law, $\int_S \underline{\mathbf{E}} \cdot \underline{\mathbf{d}S} = Q/\epsilon_0$, in conjunction with the divergence theorem to deduce the total charge enclosed by the upper hemisphere of a sphere of radius R which is centred on the origin.

Check your answer by directly evaluating the flux integral.

3) Use Gauss's Law of electrostatics in conjunction with the divergence theorem to derive the free space Maxwell equation

$$\nabla \cdot \underline{\mathbf{E}} \;=\; \frac{\rho}{\epsilon_0}$$

where ρ is the electric charge density.