PC10372, Mathematics 2
Workshop Sheet 1

Let's look at how a differential equation arises by considering a real situation and look at how the solutions to this equation behave.

Consider a population of animals, for example deer. Let $N(t)$ be the number of deer at time t.

a) Discuss why you might expect the rate of increase of N to be proportional to N. If this is the case, discuss how the deer population would change with time.

b) Clearly there must be some limits on the growth of the deer population. For example, there is a limited supply of food. Discuss why this could be modelled by making the rate of change of N proportional to $M - N$ where M is some constant number.

c) Putting together parts a) and b) above we can model the population of deer by the equation

$$\frac{dN}{dt} = \alpha\,(M - N)\,N$$

where α is a constant.

Show that this is separable and solve the equation to find

$$\frac{N}{M - N} = Ke^{M\alpha t}$$

where K is a constant. [You should find that you have to do an integral of the form $\int \frac{dN}{N(M-N)}$. Do this by partial fractions. In other words find A and B so that you can write the integral as

$$\int \frac{dN}{N(M - n)} = \int \left(\frac{A}{N} + \frac{B}{M - N}\right) dN$$

and so evaluate it.]

d) Rearrange this expression to get $N(t)$.

e) At $t = 0$ there are N_0 deer. Show that the solution satisfying this initial condition is

$$N = \frac{MN_0 e^{M\alpha t}}{M - N_0 + N_0 e^{M\alpha t}}$$

f) Adopt values $\alpha = 1$, $M = 1$. Sketch graphs of the number of deer as a function of time for the cases: i) $N_0 < M$ by choosing $N_0 = 0.1$ and ii) $N_0 > M$ by choosing $N_0 = 2$.

[For case ii) it is better to re-arrange so that $N = 2/(2 - e^{-t})$.]

g) What do your results tell you about deer populations?