Circulation

Line integrals around closed paths are given the special name 'circulation' and are written

$$\oint_C \underline{\mathbf{F}} \cdot \underline{\mathbf{d}l}$$

- the circulation of $\underline{\mathbf{F}}$ around the closed loop C. Result can depend on the path.

Example: Calculate the circulation of $\underline{\mathbf{F}}$ clockwise around a semi-circle of radius 1 centred on the origin where

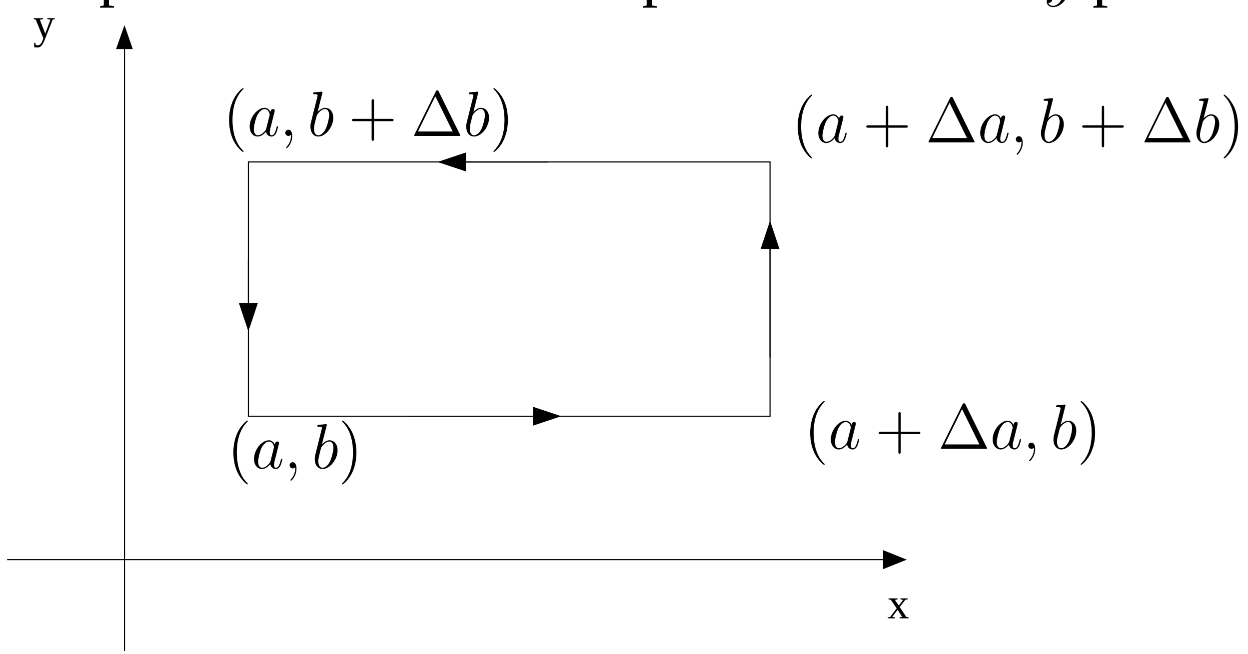
$$\underline{\mathbf{F}} = \frac{-y\underline{\mathbf{i}} + x\underline{\mathbf{j}}}{x^2 + y^2}$$

Example: Calculate the circulation of $\underline{\mathbf{A}}$ anti-clockwise around the rectangular path defined by the four points (0,0), (a,0), (a,b) and (0,b) where

$$\underline{\mathbf{A}} = (x^2 - y^2)\underline{\mathbf{i}} + 2xy\underline{\mathbf{j}}$$

The Meaning of Curl

What is the circulation of some vector field $\underline{\mathbf{v}}$ around a small rectangle lying in the plane $z = c$ which is parallel to the xy plane?



$$
\begin{aligned}
\oint_C \underline{\mathbf{v}} \cdot \underline{\mathbf{d}l} &= \oint_C [v_x(x, y, z = c)\, dx + v_y(x, y, z = c)\, dy \\
&\quad + v_z(x, y, z = c)\, dz] \\
&= \oint_C [v_x(x, y, z = c)\, dx + v_y(x, y, z = c)\, dy]
\end{aligned}
$$

as $dz = 0$ since z is constant around this path.

Consider each edge in turn:

Bottom edge, $y = b,\ dy = 0$

$$\int_a^{a+\Delta a} v_x(x, b, c)\, dx \quad = \quad \Delta a\, v_x(a, b, c)$$

plus higher order terms.

Top edge, $y = b + \Delta b,\ dy = 0$

$$\int_{a+\Delta a}^{a} v_x(x, b + \Delta b, c)\, dx \quad = \quad -\Delta a\, v_x(a, b + \Delta b, c)$$

plus higher order terms.

Combining these two edges:

$$\begin{aligned}\Delta a\,(v_x(a,b,c) - v_x(a,b+\Delta b,c)) = \\ -\Delta a\,\Delta b\frac{(v_x(a,b+\Delta b,c) - v_x(a,b,c))}{\Delta b} \\ \rightarrow \quad -\Delta a\;\;\Delta b\left.\frac{\partial v_x}{\partial y}\right|_{x=a,y=b,z=c}\end{aligned}$$

as $\Delta b \rightarrow 0$

Right edge: $x = a + \Delta a$, $dx = 0$

$$\int_b^{b+\Delta b} dy\, v_y(a+\Delta a, y, c) \;\; = \;\; \Delta b\, v_y(a+\Delta a, b, c)$$

Left edge: $x = a$, $dx = 0$

$$\int_{b+\Delta b}^{b} dy\, v_y(a, y, c) \;\; = \;\; -\Delta b\, v_y(a, b, c)$$

Combining for small Δa,

$$\Delta a\, \Delta b \frac{\partial v_y}{\partial x} \,\Big|_{x=a, y=b, z=c}$$

So combining this with the previous result,

$$\begin{aligned} \oint \underline{\mathbf{v}} \cdot \underline{\mathbf{d}l} \;\; &= \;\; \Delta a\, \Delta b \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \right) \Big|_{x=a, y=b, z=c} \\ &= \;\; \text{Area of loop times } \left(\underline{\nabla} \times \underline{\mathbf{v}} \right)_z \end{aligned}$$

But the z-component is the component in the direction normal to the plane of the loop. Let this normal be $\underline{\hat{\mathbf{n}}}$, then

$$\oint \underline{\mathbf{v}} \cdot \underline{\mathbf{d}l} \;=\; (\underline{\nabla} \times \underline{\mathbf{v}}) \cdot \underline{\hat{\mathbf{n}}}\, dA$$

were dA is the area of the elementary loop. (The direction of the normal is defined by the righthand rule.)So finally

$$\boxed{(\underline{\nabla} \times \underline{\mathbf{v}}) \cdot \underline{\hat{\mathbf{n}}} \;=\; \lim_{dA \to 0} \frac{1}{dA} \oint \underline{\mathbf{v}} \cdot \underline{\mathbf{d}l} \qquad (1)}$$

The component of the curl of a vector field in some direction $\underline{\hat{\mathbf{n}}}$ is equal to the circulation per unit area of the field around a loop to which $\underline{\hat{\mathbf{n}}}$ is the unit normal.