Circulation

Line integrals around closed paths are given the special name ‘circulation’ and are written

\[ \oint_C \mathbf{F} \cdot d\mathbf{l} \]

- the circulation of \( \mathbf{F} \) around the closed loop \( C \). Result can depend on the path.

*Example:* Calculate the circulation of \( \mathbf{F} \) clockwise around a semi-circle of radius 1 centred on the origin where

\[ \mathbf{F} = \frac{-yi + xj}{x^2 + y^2} \]

*Example:* Calculate the circulation of \( \mathbf{A} \) anti-clockwise around the rectangular path defined by the four points (0,0), (a,0), (a,b) and (0,b) where

\[ \mathbf{A} = (x^2 - y^2)i + 2xyj \]
The Meaning of Curl

What is the circulation of some vector field $\mathbf{v}$ around a small rectangle lying in the plane $z = c$ which is parallel to the $xy$ plane?

\[
\oint_C \mathbf{v} \cdot \mathbf{dl} = \oint_C \left[ v_x(x, y, z = c) \, dx + v_y(x, y, z = c) \, dy \right] + v_z(x, y, z = c) \, dz
\]

\[
= \oint_C \left[ v_x(x, y, z = c) \, dx + v_y(x, y, z = c) \, dy \right]
\]

as $dz = 0$ since $z$ is constant around this path.
Consider each edge in turn:

**Bottom edge,** \( y = b \), \( dy = 0 \)

\[
\int_{a}^{a + \Delta a} v_x(x, b, c) \, dx = \Delta a \, v_x(a, b, c)
\]

plus higher order terms.

**Top edge,** \( y = b + \Delta b \), \( dy = 0 \)

\[
\int_{a + \Delta a}^{a} v_x(x, b + \Delta b, c) \, dx = -\Delta a \, v_x(a, b + \Delta b, c)
\]

plus higher order terms.
Combining these two edges:

\[
\Delta a \left( v_x(a, b, c) - v_x(a, b + \Delta b, c) \right) = \\
-\Delta a \Delta b \frac{v_x(a, b + \Delta b, c) - v_x(a, b, c)}{\Delta b}
\]

\[
\rightarrow -\Delta a \Delta \frac{\partial v_x}{\partial y}\bigg|_{x=a,y=b,z=c}
\]

as \( \Delta b \to 0 \)
Right edge: \( x = a + \Delta a, \, dx = 0 \)
\[
\int_{b}^{b+\Delta b} dy \, v_y(a + \Delta a, y, c) = \Delta b \, v_y(a + \Delta a, b, c)
\]

Left edge: \( x = a, \, dx = 0 \)
\[
\int_{b}^{b+\Delta b} dy \, v_y(a, y, c) = -\Delta b \, v_y(a, b, c)
\]

Combining for small \( \Delta a \),
\[
\Delta a \, \Delta b \frac{\partial v_y}{\partial x} \bigg|_{x=a,y=b,z=c}
\]

So combining this with the previous result,
\[
\oint \mathbf{v} \cdot d\mathbf{l} = \Delta a \, \Delta b \left( \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \right) \bigg|_{x=a,y=b,z=c}
\]
\[
= \text{Area of loop times} \, (\nabla \times \mathbf{v})_z
\]
But the $z$-component is the component in the direction normal to the plane of the loop. Let this normal be $\hat{n}$, then

$$\oint v \cdot dl = (\nabla \times v) \cdot \hat{n} \, dA$$

were $dA$ is the area of the elementary loop. (The direction of the normal is defined by the righthand rule.) So finally

$$\lim_{dA \to 0} \frac{1}{dA} \oint v \cdot dl$$  \hspace{1cm} (1)

The component of the curl of a vector field in some direction $\hat{n}$ is equal to the circulation per unit area of the field around a loop to which $\hat{n}$ is the unit normal.