Continuity Equation

Consider the flow of material with density ρ with velocity $\underline{\mathbf{v}}$ into some volume $d\tau$.

The mass of gas in the volume is $\rho\, d\tau$ and the net rate of increase of mass in the volume per unit time is $\frac{\partial \rho}{\partial t}\, d\tau$.

What mass of material flows in to the volume through its surface?

The mass current $\underline{\mathbf{j}} = \rho \underline{\mathbf{v}}$ (mass per unit time per unit area)

but integrated over the whole surface this must just equal the change in mass within the volume, so

$$\int_{\text{surface}} \underline{\mathbf{j}} \cdot \underline{dA} = \int_{\text{surface}} \rho \underline{\mathbf{v}} \cdot \underline{dA}$$

$$= \underline{\nabla} \cdot (\rho \underline{\mathbf{v}})\, d\tau = -\frac{\partial \rho}{\partial t}\, d\tau$$

Continuity Eqn., cont.

$$\boxed{\begin{aligned} \underline{\nabla} \cdot (\rho \underline{\mathbf{v}}) &= -\frac{\partial \rho}{\partial t} \\ \underline{\nabla} \cdot \left(\underline{\mathbf{j}}\right) &= -\frac{\partial \rho}{\partial t} \end{aligned}}$$

- continuity equation - conserved quantity - mass or electric charge (ρ charge density (C m^{-3}), $\underline{\mathbf{j}}$ current density (C $\mathrm{m}^{-2}\mathrm{s}^{-1}$))

The Divergence Theorem

Consider a closed surface S enclose a volume V. Divide it into 2 parts S_1 and S_2. Note that $S = S_1 + S_2$ where S_1 and S_2 are open surfaces by $S_1 + S_2$ is a closed surface. Let the connecting surface that divides the volume in to V_1 and V_2 be S_c.
Consider the total flux, F, of the field $\underline{\mathbf{v}}$ out of S,

$$F = \int_S \underline{\mathbf{v}} \cdot \underline{\mathbf{d}A} = \int_{S_1+S_2} \underline{\mathbf{v}} \cdot \underline{\mathbf{d}A}$$

Consider V_1. The normal to S_c to be out of V_1 (and into V_2).But considering volume V_2, the normal to S_c is out of V_2 (and into V_1).
Flux out of volume V_1, F_1 is

$$F_1 = \int_{S_1} \underline{\mathbf{v}} \cdot \underline{\mathbf{d}A} + \int_{S_c 1} \underline{\mathbf{v}} \cdot \underline{\mathbf{d}A}_1$$

and similarly the flux out of volume V_2, F_2 is

$$F_2 = \int_{S_2} \underline{\mathbf{v}} \cdot \underline{\mathbf{d}A} + \int_{S_c 2} \underline{\mathbf{v}} \cdot \underline{\mathbf{d}A}_2$$

But clearly $\underline{\mathbf{d}A}_1 = -\underline{\mathbf{d}A}_2$ and

$$\int_{S_c1} \underline{\mathbf{v}} \cdot \underline{\mathbf{d}A}_1 = -\int_{S_c2} \underline{\mathbf{v}} \cdot \underline{\mathbf{d}A}_1$$

Hence

$$F_1 + F_2 \quad = \quad F = \int_{S_1} \underline{\mathbf{v}} \cdot \underline{\mathbf{d}A} + \int_{S_2} \underline{\mathbf{v}} \cdot \underline{\mathbf{d}A}$$

So the flux across internal surfaces cancel. What flows out of one flows into the next.

Now imagine dividing the volume V in to infinitesimal volumes, labelled by index i. So the flux out of the volume is F where

$$\begin{aligned} F &= \int_S \underline{\mathbf{v}} \cdot \underline{\mathbf{d}A} \\ &= \sum_i \int_{S_i} \underline{\mathbf{v}} \cdot \underline{\mathbf{d}A} = \sum_i \underline{\nabla} \cdot \underline{\mathbf{v}}\, dV_i \end{aligned}$$

using our integral definition of the divergence. In the limit $dV_i \to 0$, the sum becomes an integral and

$$\boxed{\int_S \underline{\mathbf{v}} \cdot \underline{\mathbf{d}A} \quad = \quad \int_V \underline{\nabla} \cdot \underline{\mathbf{v}}\, dV}$$

which is the divergence theorem.

Examples:

1) $\underline{\mathbf{v}} = x^2\underline{\mathbf{i}} + y^2\underline{\mathbf{j}} + z^2\underline{\mathbf{k}}$ from a cube.
2) $\underline{\mathbf{v}} = z^3\underline{\mathbf{k}}$ out of a sphere
3) Verify the divergence theorem for $\underline{\mathbf{F}} = r^2\hat{\underline{\mathbf{r}}}$ for a sphere of radius R centred on the origin.
4) Divergence in cylindrical polar coordinates.