Flux Integrals

Flux integral: Calculate flow out of (or in to) a volume.

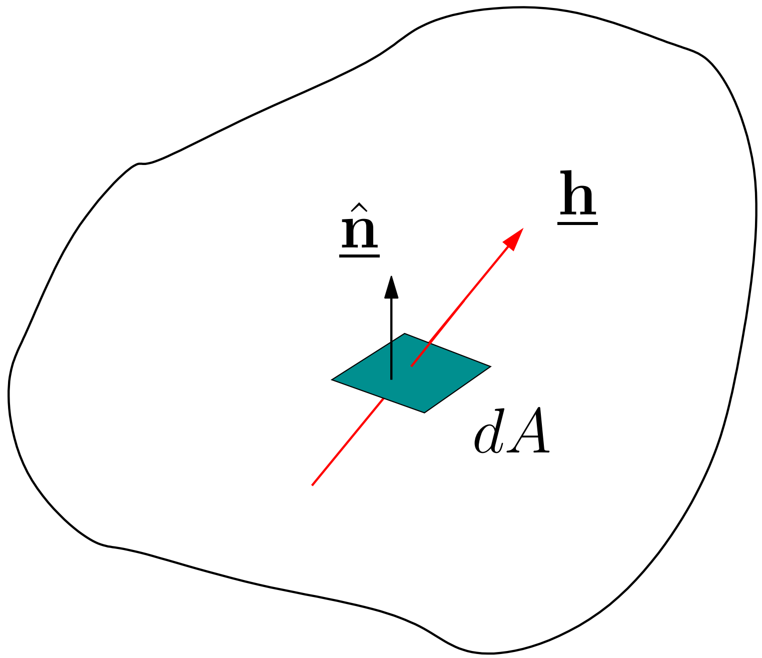
Closed surface: a surface fully enclosing a volume. A surface which separates an inside from an outside.

Consider a small patch of area dA on the surface and a unit vector $\underline{\hat{\mathbf{n}}}$ normal (pointing outwards) to the patch of surface.

If $\underline{\mathbf{h}}$ is some vector field, for example heat current - heat flow per unit area (J s^{-1} m^{-2}).

How much heat flows out of the surface per unit time ?

Rate of heat flow across the elementary area $= \underline{\mathbf{h}} \cdot \underline{\hat{\mathbf{n}}}\, dA$



Flux Integrals, cont.

Typically write $\underline{dA} = \hat{\underline{\mathbf{n}}}\, dA$ so

$$H = \int_{\text{surface}} \underline{dA}$$

- $\underline{dA}$ is a vector in the direction of the outward normal to surface with magnitude equal to the element of area.

The concept of a flux integral can also be extended to an open surface (but need to define direction of normal).

$$\boxed{\text{Flux} = \int_S \underline{\mathbf{v}} \cdot \underline{dA}}$$

Examples:

1) Calculate flux of field $\underline{\mathbf{v}} = x^2\underline{\mathbf{i}} + y^2\underline{\mathbf{j}} + z^2\underline{\mathbf{k}}$ out of cube of side length l with $0 < x < l$, $0 < y < l$ and $0 < z < l$.

2) Calculate the flux of $\underline{\mathbf{v}} = z^3\underline{\mathbf{k}}$ out of a sphere of radius R centred on the origin.

Divergence Theorem

Consider a small box with one corner at (a, b, c) and side $\Delta a, \Delta b, \Delta c$. Calculate the flux of the general vector field $\underline{\mathbf{v}}$ out of this volume where

$$\underline{\mathbf{v}}(x, y, z) = v_x(x, y, z)\underline{\mathbf{i}} + v_y(x, y, z)\underline{\mathbf{j}} + v_z(x, y, z)\underline{\mathbf{k}}$$

Need to consider all six faces separately:
1) Top face, $z = c + \Delta c$

$$\begin{aligned} \underline{dA} &= dx\,dy\,\underline{\mathbf{k}} \\ \underline{dA} \cdot \underline{\mathbf{v}}(x, y, c + \Delta c) &= dx\,dy\,v_z(x, y, c + \Delta c) \\ \therefore \text{ Flux out of top face, } F_T &= \int_a^{a+\Delta a} dx \int_b^{b+\Delta b} dy\,v_z(x, y, c + \Delta c) \end{aligned}$$

Since the box is infinitesimally small, x and y are nearly constant over the box so can replace x with a and y with b when $\Delta a, \Delta b \to 0$, so

$$F_T = \Delta a\,\Delta b\,v_z(a, b, c + \Delta c)$$

2) Bottom face, $z = c$

$$\begin{aligned}
\underline{dA} &= dx\,dy\,(-\underline{\mathbf{k}}) \\
\underline{dA} \cdot \underline{\mathbf{v}}(x, y, c + \Delta c) &= -dx\,dy\,v_z(x, y, c) \\
\therefore \text{Flux out of bottom face, } F_B &= -\int_a^{a+\Delta a} dx \int_b^{b+\Delta b} dy\,v_z(x, y, c) \\
F_B &= -\Delta a\,\Delta b\,v_z(a, b, c)
\end{aligned}$$

So combining the top and bottom faces:

$$\begin{aligned}
F_T + F_B &= \Delta a\,\Delta b\,(v_z(a, b, c + \Delta c) - v_z(a, b, c)) \\
&= \Delta a\,\Delta b\,\Delta c \left(\frac{v_z(a, b, c + \Delta c) - v_z(a, b, c)}{\Delta c} \right)
\end{aligned}$$

But as $\Delta c \to 0$

$$() \to \left. \frac{\partial v_z}{\partial z} \right|_{x=a, y=b, z=c}$$

and so

$$F_T + F_B \;=\; \Delta a\,\Delta b\,\Delta c\,\left.\frac{\partial v_z}{\partial z}\right|_{x=a,y=b,z=c}$$

$\Delta a\,\Delta b\,\Delta c$ - volume of box

3) Left face, $x = a$, F_L

$$\begin{aligned} \underline{dA} &= dy\,dz\,(-\underline{\mathbf{i}}) \\ \underline{dA}\cdot\underline{\mathbf{v}}(a,y,z) &= dy\,dz\,v_x(a,y,z) \end{aligned}$$

4) Right face, $x = a + \Delta a$, F_R

$$\rightarrow F_L + F_R \;=\; \Delta a\,\Delta b\,\Delta c\,\left.\frac{\partial v_x}{\partial x}\right|_{x=a,y=b,z=c}$$

5) Back face, $y = b$, F_{Bk}
6) Front face, $y = b + \Delta y$, F_F

$$\rightarrow F_{Bk} + F_F \quad = \quad \Delta a \, \Delta b \, \Delta c \, \frac{\partial v_y}{\partial y}\bigg|_{x=a, y=b, z=c}$$

$$\begin{aligned} \text{Total flux} \quad &= \quad \Delta a \, \Delta b \, \Delta c \left[\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \right]_{x=a, y=b, z=c} \\ &= \quad \Delta a \, \Delta b \, \Delta c \; \underline{\nabla} \cdot \underline{\mathbf{v}}\big|_{x=a, y=b, z=c} \end{aligned}$$

The divergence of a vector field at a point P is equal to the flux ('outflow') of the field per unit volume at the point P.

$$\boxed{\underline{\nabla} \cdot \underline{\mathbf{v}} \quad = \quad \lim_{\delta\tau \to 0} \frac{1}{\delta\tau} \int_{\text{surface}} \underline{\mathbf{v}} \cdot \underline{dA}}$$

Note: 1) $\underline{\nabla} \cdot \underline{\mathbf{v}} < 0$ - inflow

2) For incompresible fluids, the velocity field $\underline{\mathbf{v}}$ must satisfy $\underline{\nabla} \cdot \underline{\mathbf{v}} = 0$ - what flows in must flow out.