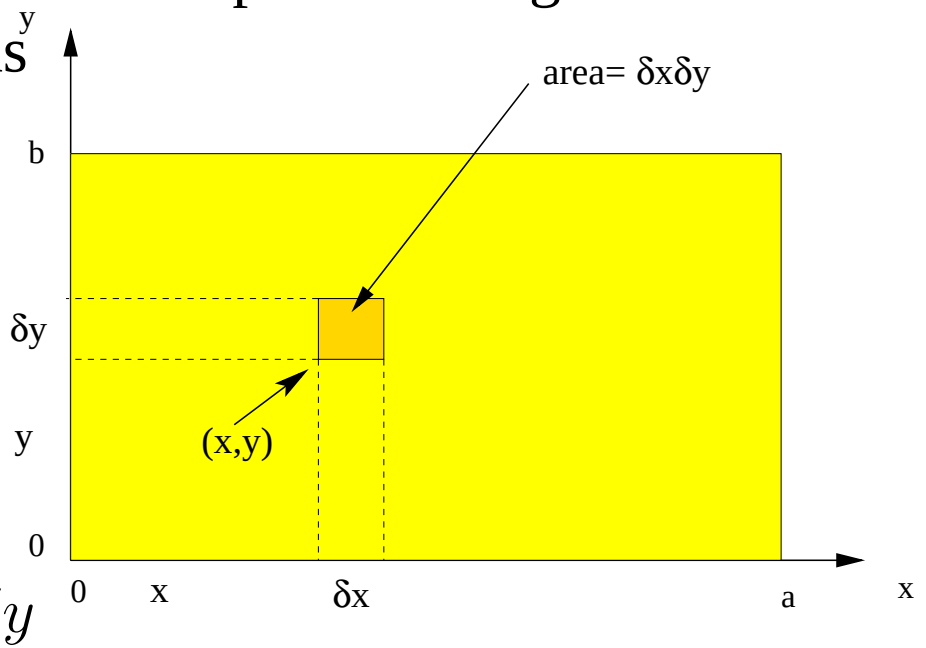
Multiple Integration

What is the mass of this rectangular plate? Density, $\rho(x,y) = Cxy$, (C is a constant.)

Divide plate into elementary rectangles, and add up the rectangles

Each small rectangle at position (x,y) has mass δm where

$$\delta m \;=\; \rho(x,y)\,\delta x\,\delta y$$

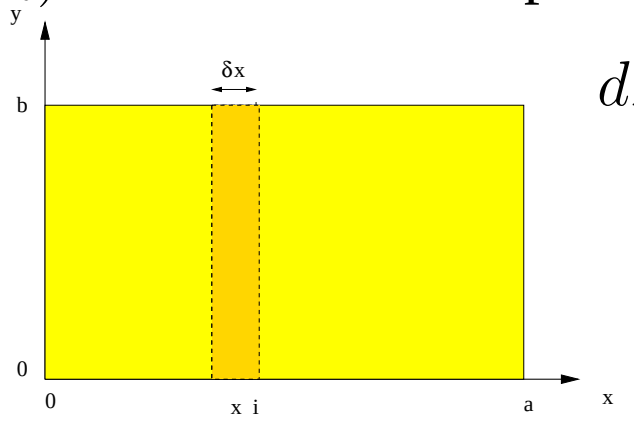


Total mass M

$$M \;=\; \lim_{\delta x,\delta y\to 0} \sum_{rectangles} \rho(x,y)\,\delta x\,\delta y$$

(This limit is just an integral over both x and y.)

Consider keeping the value of x fixed, and summing first over y, given mass $dM(x_i)$ in the vertical strip

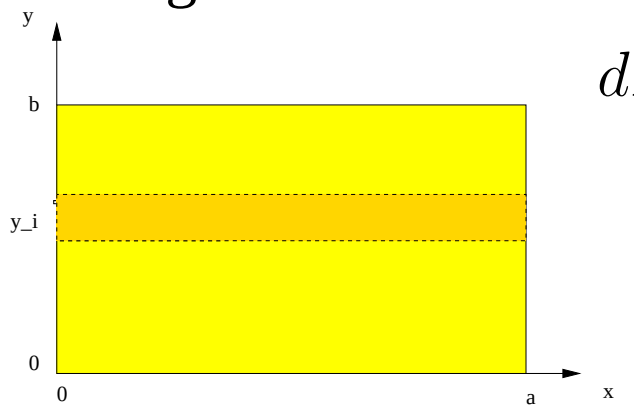


$$\begin{aligned} dM(x_i) &= \delta x \int_0^b dy\, \rho(x,y) \\ &= \delta x \int_0^b dy\, C x_i y \\ &= \delta x\, C x_i \int_0^b dy\, y = \delta x\, C x_i \left(\frac{1}{2} b^2\right) \end{aligned}$$

Now sum up all the columns:

$$\begin{aligned} M &= \lim_{\delta x \to 0} \sum_i dM(x_i) = \lim_{\delta x \to 0} \sum_i \delta x\, C x_i \left(\frac{1}{2} b^2\right) \\ &= C \left(\frac{1}{2} b^2\right) \int_0^a x\, dx = C \left(\frac{b^2}{2}\right) \left(\frac{a^2}{2}\right) \end{aligned}$$

Could have got the same answer by holding y fixed first:



$$\begin{aligned} dM(y_i) &= \delta y \int_0^a dx\, Cxy_i \\ &= Cy_i \delta y \left(\frac{a^2}{2}\right) \end{aligned}$$

Now sum over rows:

$$\begin{aligned} M &= \lim_{\delta y \to 0} \sum_i dM(y_i) = \lim_{\delta y \to 0} \sum_i \delta y\, Cy_i \left(\frac{a^2}{2}\right) = C\left(\frac{a^2}{2}\right) \int_0^b dy\, y \\ &= C\left(\frac{b^2}{2}\right)\left(\frac{a^2}{2}\right) \end{aligned}$$

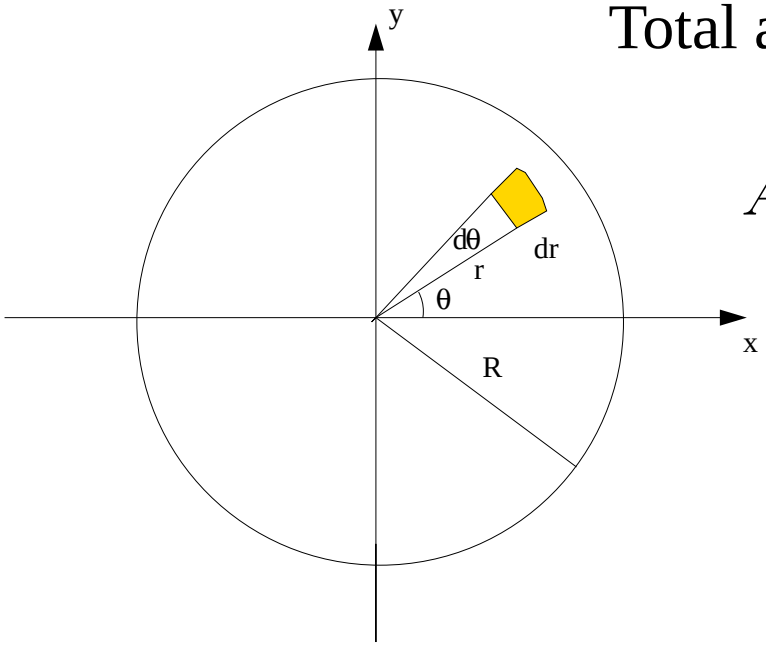
The mass was given by the double integration:

$$\begin{aligned} M &= \int_0^a dx \left(\int_0^b dy\, \rho(x,y) \right) \\ &= \int_0^b dy \left(\int_0^a dx\, \rho(x,y) \right) \end{aligned}$$

Example: Mass of plate $2 < x < 4$, $0 < y < 1$ with density $\rho(x,y) = xy + y^2 - 1$

Example: Area of circle

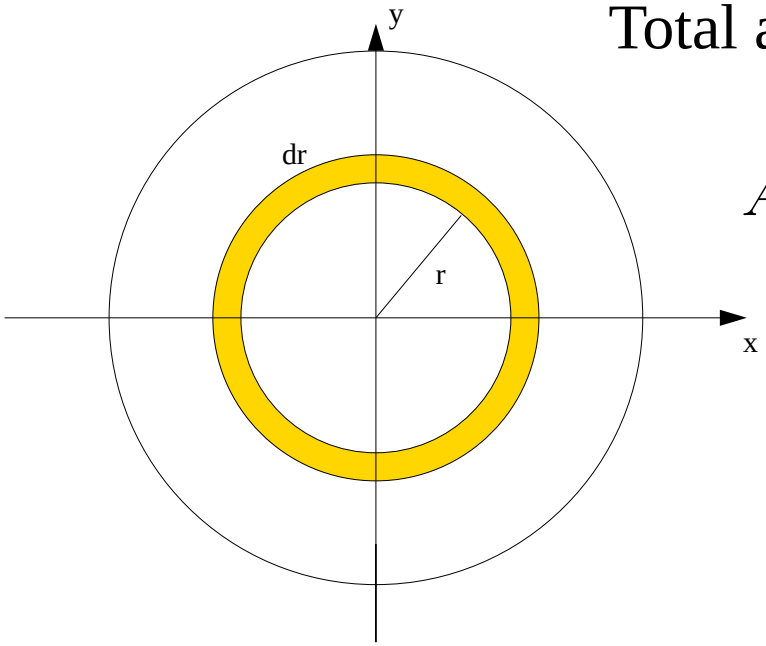
Area of element $dA = r\,dr\,d\theta$



Total area, A

$$A = \int_{circle} dA = \int_0^{2\pi} d\theta \int_0^R r\,dr$$

If and only if the function to be integrated is circularly symmetry, can use Area of element $dA = 2\pi r\, dr$



Total area, A

$$A = \int_{circle} dA = 2\pi \int_0^R r\, dr$$