Second Derivatives

1. $\underline{\nabla} \cdot (\underline{\nabla} T)$ - scalar

$$= \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right) = \underline{\nabla}^2 T$$

$$\rightarrow \underline{\nabla}^2 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \text{ Laplacian}$$

2. $\underline{\nabla} \times (\underline{\nabla} T)$ - vector

$$\text{curl grad}\, T = \underline{\nabla} \times (\underline{\nabla} T) = 0$$

3. $\underline{\nabla}(\underline{\nabla} \cdot \underline{\mathbf{v}})$ - vector

4. $\underline{\nabla} \cdot (\underline{\nabla} \times \underline{\mathbf{v}})$ - vector

$$\text{div curl}\, \underline{\mathbf{v}} = 0$$

Second Derivatives, cont.

5. $\underline{\nabla} \times (\underline{\nabla} \times \underline{\mathbf{v}})$ - vector
   Recall $\underline{\mathbf{a}} \times (\underline{\mathbf{b}} \times \underline{\mathbf{c}}) = \underline{\mathbf{b}}(\underline{\mathbf{a}} \cdot \underline{\mathbf{c}}) - (\underline{\mathbf{a}} \cdot \underline{\mathbf{b}})\underline{\mathbf{c}}$ so

$$\begin{aligned}
\underline{\nabla} \times (\underline{\nabla} \times \underline{\mathbf{v}}) &= \underline{\nabla}(\underline{\nabla} \cdot \underline{\mathbf{v}}) - (\underline{\nabla} \cdot \underline{\nabla})\underline{\mathbf{v}} \\
&= \underline{\nabla}(\underline{\nabla} \cdot \underline{\mathbf{v}}) - \underline{\nabla}^2\underline{\mathbf{v}}
\end{aligned}$$

Summary

1. $\underline{\nabla} \cdot (\underline{\nabla} T) = \text{div grad T} = \underline{\nabla}^2 T$
2. $\underline{\nabla} \times (\underline{\nabla} T) = \text{curl grad T} = 0$
3. $\underline{\nabla} \cdot (\underline{\nabla} \times \underline{\mathbf{v}}) = \text{div curl}\, \underline{\mathbf{v}} = 0$
4. $\underline{\nabla} \times (\underline{\nabla} \times \underline{\mathbf{v}}) = \underline{\nabla}(\underline{\nabla} \cdot \underline{\mathbf{v}}) - \underline{\nabla}^2 \underline{\mathbf{v}}$
5. $\underline{\nabla} \cdot (\phi \underline{\mathbf{v}}) = \phi \underline{\nabla} \cdot \underline{\mathbf{v}} + \underline{\mathbf{v}} \cdot \underline{\nabla} \phi$
6. $\underline{\nabla} \times (\phi \underline{\mathbf{v}}) = \phi \underline{\nabla} \times \underline{\mathbf{v}} + \underline{\nabla} \phi \times \underline{\mathbf{v}}$
7. $\underline{\nabla} \cdot (\underline{\mathbf{v}} \times \underline{\mathbf{w}}) = (\underline{\nabla} \times \underline{\mathbf{v}}) \cdot \underline{\mathbf{w}} - (\underline{\nabla} \times \underline{\mathbf{w}}) \cdot \underline{\mathbf{v}}$
8. $\underline{\nabla} \times (\underline{\mathbf{v}} \times \underline{\mathbf{w}}) = (\underline{\nabla} \cdot \underline{\mathbf{w}}) \underline{\mathbf{v}} + (\underline{\mathbf{w}} \cdot \underline{\nabla}) \underline{\mathbf{v}} - (\underline{\nabla} \cdot \underline{\mathbf{v}}) \underline{\mathbf{w}} - (\underline{\mathbf{v}} \cdot \underline{\nabla}) \underline{\mathbf{w}}$

Cylindrical Polar Coordinates

$$
\begin{aligned}
\underline{\nabla} \cdot \underline{\mathbf{v}} &= \frac{1}{r}\frac{\partial}{\partial r}(r v_r) + \frac{1}{r}\frac{\partial}{\partial \theta} v_\theta + \frac{\partial}{\partial z} v_z \\
\underline{\nabla} \times \underline{\mathbf{v}} &= \frac{1}{r}\begin{vmatrix} \hat{\underline{\mathbf{r}}} & r\hat{\underline{\boldsymbol{\theta}}} & \underline{\mathbf{k}} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ v_r & r v_\theta & v_z \end{vmatrix} \\
\underline{\nabla}^2 f &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2}
\end{aligned}
$$

Spherical Polar Coordinates

$$\underline{\nabla} \cdot \underline{\mathbf{v}} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 v_r\right) + \frac{1}{r\,\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, v_\theta) + \frac{1}{r\,\sin\theta}\frac{\partial}{\partial\phi}(v_\phi)$$

$$\underline{\nabla} \times \underline{\mathbf{v}} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \underline{\hat{\mathbf{r}}} & r\underline{\hat{\boldsymbol{\theta}}} & r\,\sin\theta\,\underline{\hat{\boldsymbol{\phi}}} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ v_r & r v_\theta & r\,\sin\theta\, v_\phi \end{vmatrix}$$

$$\underline{\nabla}^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial^2}{\partial\theta^2}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}$$

Summary

1. $\underline{\nabla} \cdot (\underline{\nabla} T) = \text{div grad T} = \underline{\nabla}^2 T$
2. $\underline{\nabla} \times (\underline{\nabla} T) = \text{curl grad T} = 0$
3. $\underline{\nabla} \cdot (\underline{\nabla} \times \underline{\mathbf{v}}) = \text{div curl}\,\underline{\mathbf{v}} = 0$
4. $\underline{\nabla} \times (\underline{\nabla} \times \underline{\mathbf{v}}) = \underline{\nabla}(\underline{\nabla} \cdot \underline{\mathbf{v}}) - \underline{\nabla}^2 \underline{\mathbf{v}}$
5. $\underline{\nabla} \cdot (\phi \underline{\mathbf{v}}) = \phi \underline{\nabla} \cdot \underline{\mathbf{v}} + \underline{\mathbf{v}} \cdot \underline{\nabla} \phi$
6. $\underline{\nabla} \times (\phi \underline{\mathbf{v}}) = \phi \underline{\nabla} \times \underline{\mathbf{v}} + \underline{\nabla} \phi \times \underline{\mathbf{v}}$
7. $\underline{\nabla} \cdot (\underline{\mathbf{v}} \times \underline{\mathbf{w}}) = (\underline{\nabla} \times \underline{\mathbf{v}}) \cdot \underline{\mathbf{w}} - (\underline{\nabla} \times \underline{\mathbf{w}}) \cdot \underline{\mathbf{v}}$
8. $\underline{\nabla} \times (\underline{\mathbf{v}} \times \underline{\mathbf{w}}) = (\underline{\nabla} \cdot \underline{\mathbf{w}}) \underline{\mathbf{v}} + (\underline{\mathbf{w}} \cdot \underline{\nabla}) \underline{\mathbf{v}} - (\underline{\nabla} \cdot \underline{\mathbf{v}}) \underline{\mathbf{w}} - (\underline{\mathbf{v}} \cdot \underline{\nabla}) \underline{\mathbf{w}}$