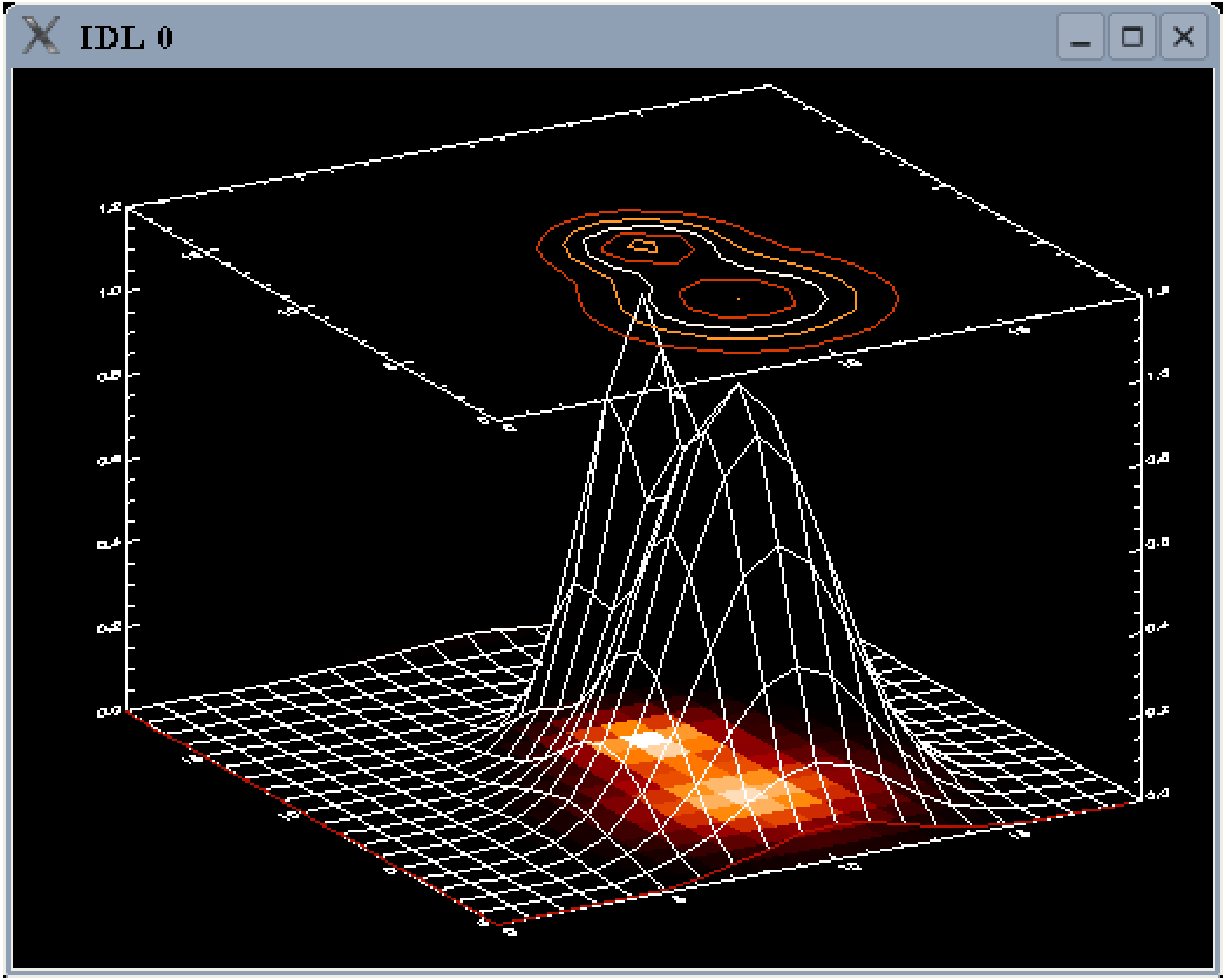
Fields

A field is a quantity which varies with position.
This quantity can be a single number, a scalar, in which case the field is a scalar field e.g. $T(x, y, z)$
Examples: temperature, altitude of land, density
Alternatively the quantity can be a vector, in which case the field is a vector field
Example vector fields: velocity of a fluid, electric field
At each point in space the quantity has both a magnitude and a direction.
The value of a field at a given point does not depend on the coordinate system.

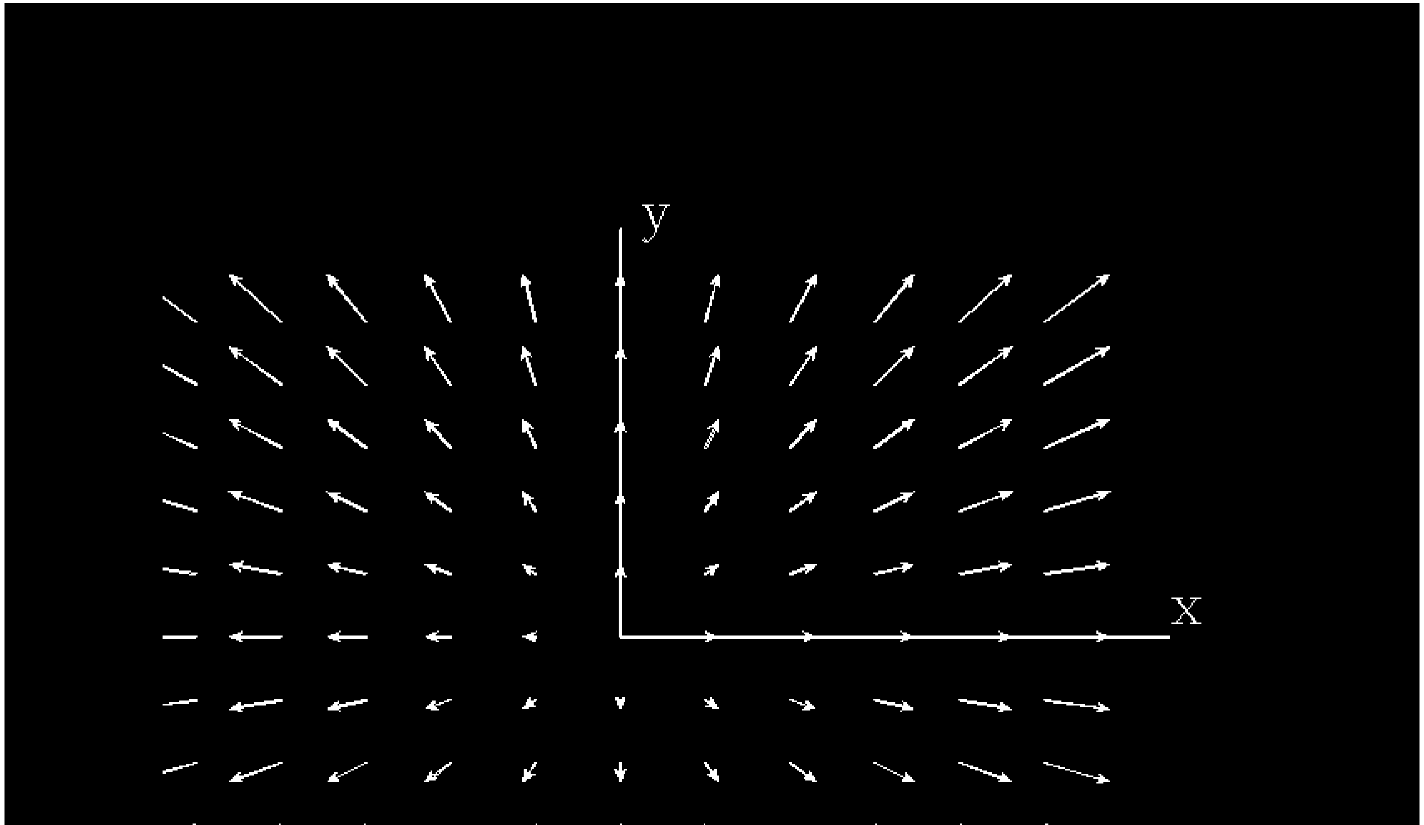
Three different representations of a scalar field:



Example vector field:

$$\underline{v}(x,y) \quad = \quad \hat{\underline{\mathbf{r}}} = \frac{x\underline{i} + y\underline{j}}{\left(x^2+y^2\right)^{1/2}}$$

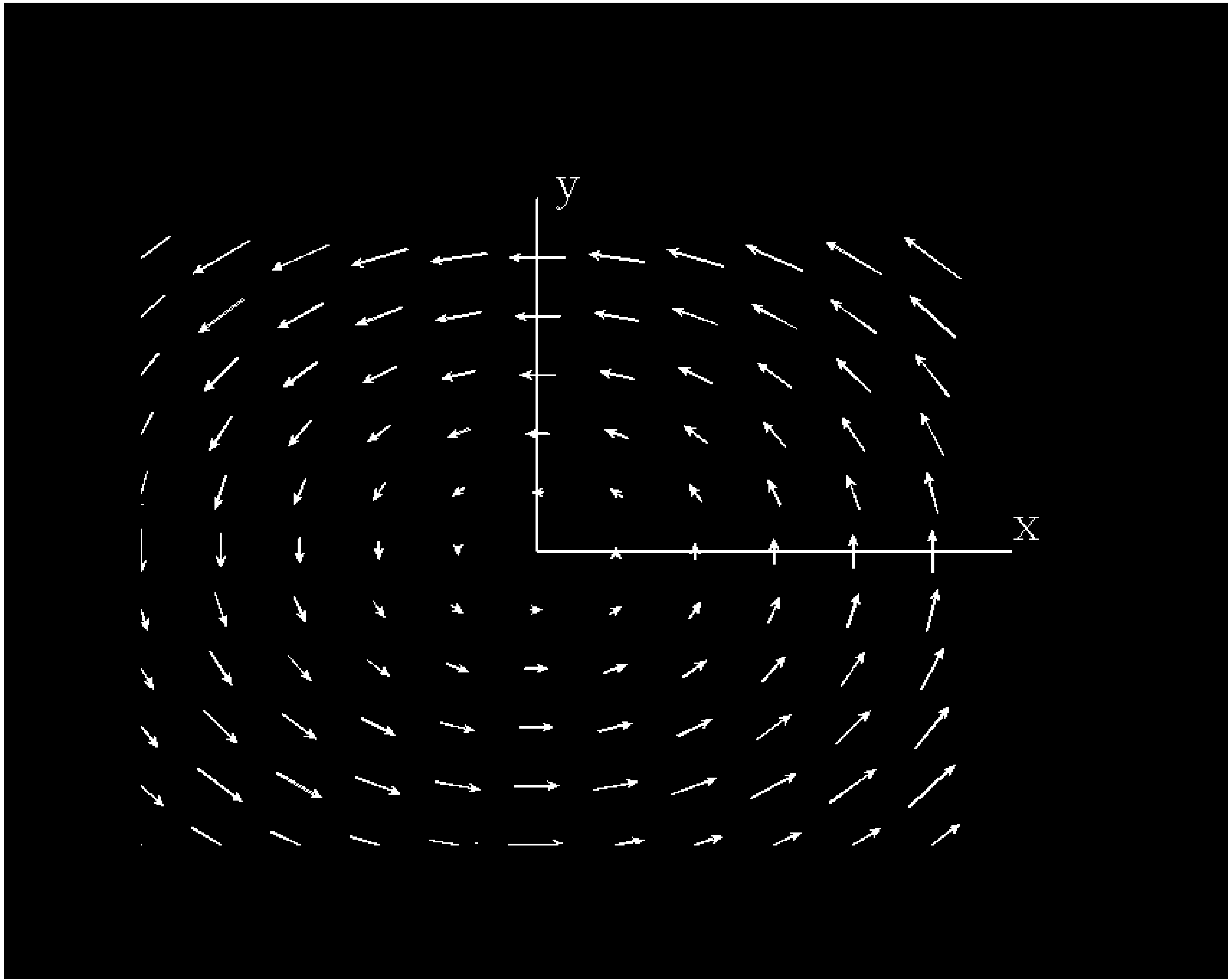
$$\underline{r} = x\underline{i} + y\underline{j} \qquad\qquad r = |\underline{r}| = \sqrt{x^2+y^2},\ \underline{\hat{r}} = \frac{\underline{r}}{r}$$



Diverging vector field

$$\underline{v}(x,y) = \frac{-y\underline{i}+x\underline{j}}{(x^2+y^2)^{3/2}}$$

Curling vector field



Field Lines

Field lines show the direction of a vector field, but they don't show any information about the magnitude.

How to Calculate the field lines

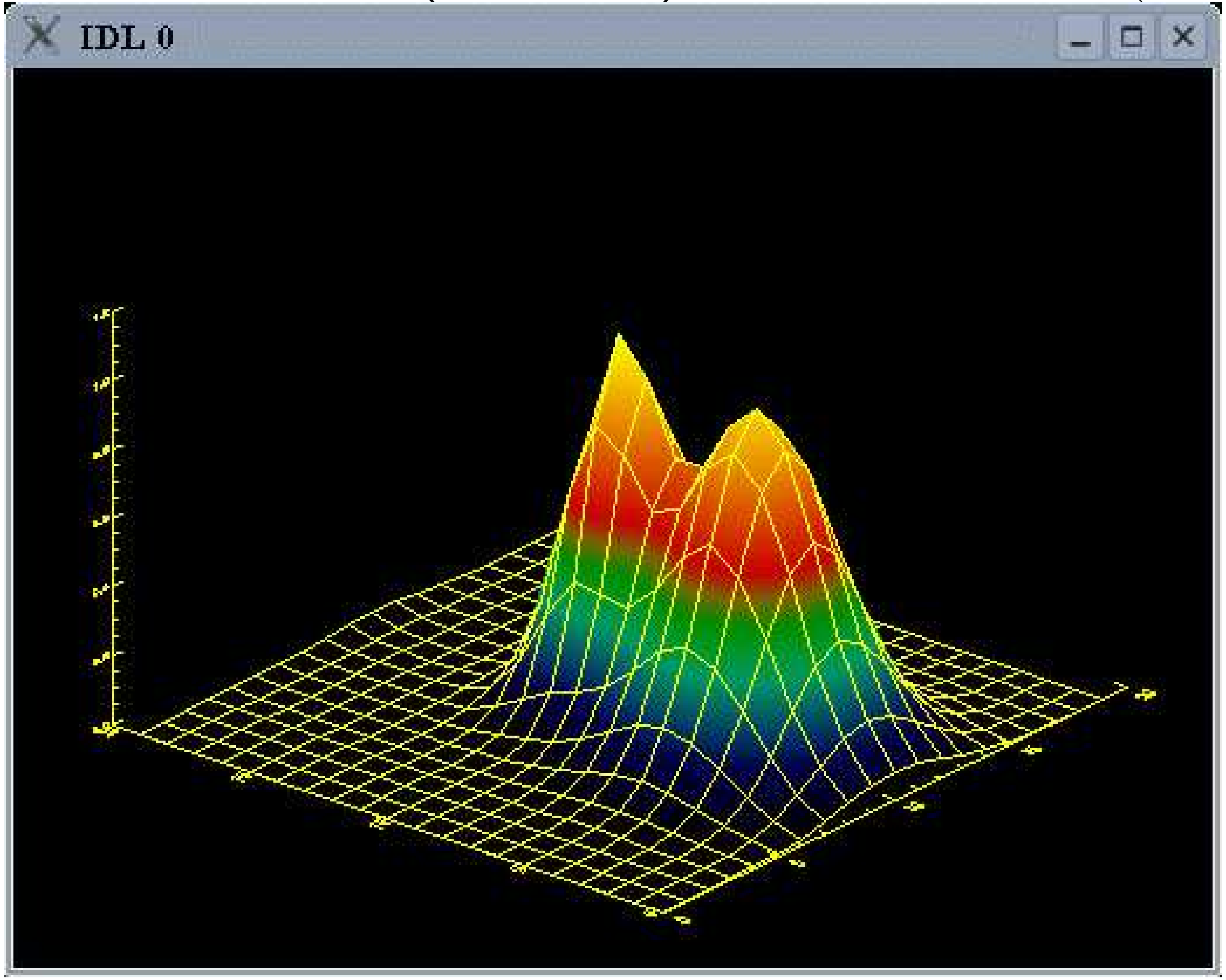
Write a 2-d field as $\underline{v}(x,y) = v_x(x,y)\underline{i} + v_y(x,y)\underline{j}$.
The tanget to the field line makes an angle θ to the x-axis where $\tan\theta = \dfrac{v_y(x,y)}{v_x(x,y)}$.
Let the field line be $y = f(x)$, but on this field line the slope of the tanget to the field line is $\dfrac{dy}{dx} = \dfrac{v_y(x,y)}{v_x(x,y)}$, which we can integrate to find the equation to the field line.
Example: Calculate the field lines of the vector field

$$\underline{v}(x,y) = \frac{-y\underline{i} + x\underline{j}}{(x^2+y^2)^{3/2}}$$

Field lines: $x^2 + y^2 = c^2$

The Gradient (Grad)

What's the slope (derivative) of a scalar field $H(x, y)$?



Consider the temperature field $T(x, y, z)$ at the two points (x, y, z) and $(x + dx, y + dy, z + dz)$.
The difference in temperature between these two points is dT where

$$dT = \frac{\partial T}{\partial x} dx + \frac{\partial T}{\partial y} dy + \frac{\partial T}{\partial z} dz$$

But the vector between these two points is $\underline{\mathbf{dr}} = dx\, \underline{i} + dy\, \underline{j} + dz\, \underline{k}$. So we can write

$$\begin{aligned} dT &= \left(\underline{i} \frac{\partial T}{\partial x} + \underline{j} \frac{\partial T}{\partial y} + \underline{k} \frac{\partial T}{\partial z} \right) \cdot \left(dx\, \underline{i} + dy\, \underline{j} + dz\, \underline{k} \right) \\ dT &= \underline{\nabla} T \cdot \underline{\mathbf{dr}} \end{aligned}$$

where

$$\underline{\nabla} T = \underline{i} \frac{\partial T}{\partial x} + \underline{j} \frac{\partial T}{\partial y} + \underline{k} \frac{\partial T}{\partial z}$$

which is a **vector field** called the gradient, or grad of the scalar field T.
Also often written as *grad* T.

Expect to see the gradient of scalar fields in various physical laws, e.g. heat flow in 1-D $h = -k\dfrac{\Delta T}{\Delta x}$ while in 3-D $\underline{h} = -k\underline{\nabla}T$

Heat flows in the direction of $-$grad T, the direction of decreasing T and normal to the isotherms.

Properties of The Gradient

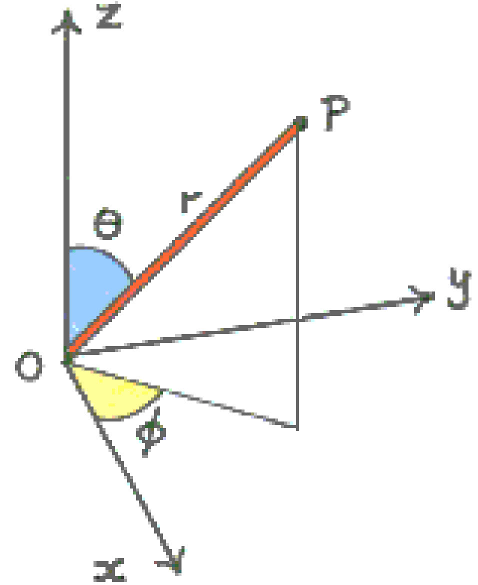
0) $T(x,y,z)$ is a scalar field
1) $\underline{\nabla} T(x,y,z)$ is a vector field
2) $dT = T(\underline{r} + \underline{\mathbf{dr}}) - T(r) = \underline{\nabla} T \cdot \underline{\mathbf{dr}}$ is the change in T between points $\underline{r}$ and $\underline{r} + \underline{dr}$
If $\underline{\hat{\mathbf{u}}}$ is a unit vector parallel to $\underline{\mathbf{dr}}$ so $\underline{\mathbf{dr}} = \underline{\hat{u}}\, ds$ $(ds = |\underline{\mathbf{dr}}|)$ then
$dT = \underline{\nabla} T \cdot \underline{\hat{u}}\, ds$ so
3) $\left.\dfrac{dT}{ds}\right|_{\underline{\hat{u}}} = \underline{\nabla} T \cdot \underline{\hat{u}}$ is the *directional derivative* - the rate of change of T in the direction of the unit vector $\underline{\hat{u}}$
4) $\underline{\nabla} T \cdot \underline{\hat{u}}$ is a maximum when $\underline{\nabla} T$ is parallel to $\underline{\hat{u}}$
5) $|\underline{\nabla} T|$ is the maximum rate of change and it is in the direction $\underline{\nabla} T$
Consider the surface $T =$constant. Any point P on this surface $\left.\dfrac{dT}{ds}\right|_{\underline{\hat{u}}} = 0$ for any vector u tangent to the surface at P. i.e. $\underline{\nabla} T \cdot \left.\dfrac{dT}{ds}\right|_{\underline{\hat{u}}} = 0$ so
6) $\underline{\nabla} T$ is normal to the surface of constant T.

Coordinate Systems

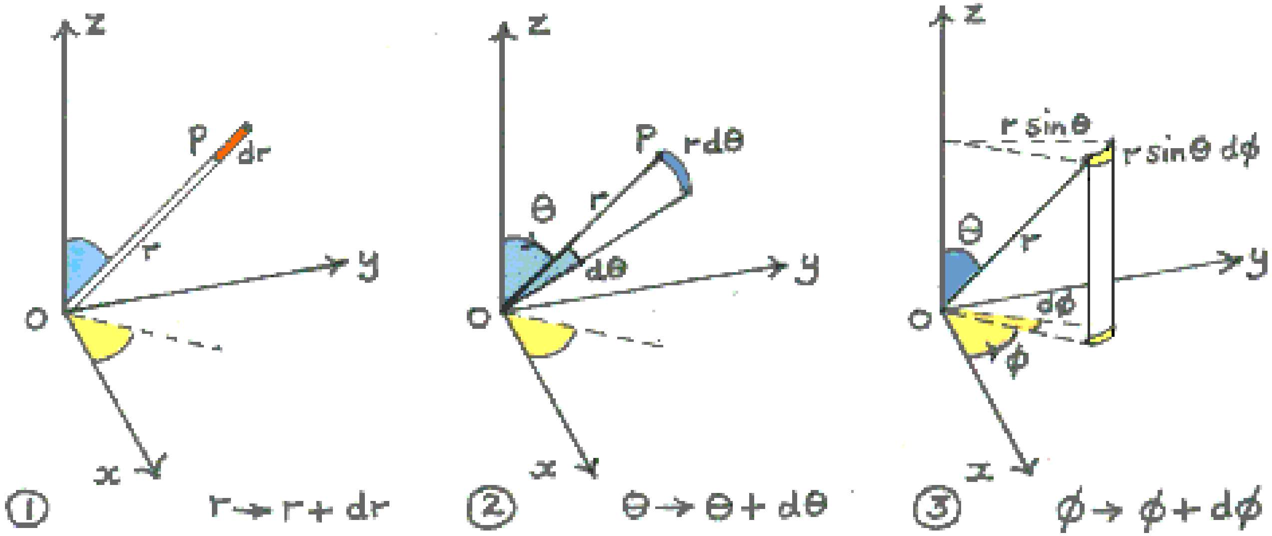
Often easier to solve problems in coordinates other that Cartesian e.g. spherical polar coordinates for spherically symmetric systems or for systems with cylindrical symmetry, cylindrical polar coordinates.
Define $d\underline{r}$ as an infinitesimal displacement.
Cartesian Coordinates

$$\begin{aligned} d\underline{r} &= dx\,\underline{i} + dy\,\underline{j} + dz\,\underline{k} \\ \text{and} \quad \underline{\nabla} T &= \underline{i}\frac{\partial T}{\partial x} + \underline{j}\frac{\partial T}{\partial y} + \underline{k}\frac{\partial T}{\partial z} \end{aligned}$$

Spherical Polar Coordinates



Unit vectors are $\underline{\hat{r}}$, $\underline{\hat{\theta}}$, and $\underline{\hat{\phi}}$



$$d\underline{r} \;=\; dr\,\underline{\hat{r}} + r\,d\theta\,\underline{\hat{\theta}} + r\sin\theta\,d\phi\,\underline{\hat{\phi}}$$

Can expand $\underline{\nabla}T$ as $\underline{\nabla}T = a_r\,\underline{\hat{r}} + a_\theta\,\underline{\hat{\theta}} + a_\phi\,\underline{\hat{\phi}}$

So $dT = \underline{\nabla}T \cdot d\underline{r} = a_r\,dr + r a_\theta\,d\theta + r\sin\theta\,a_\phi\,d\phi$

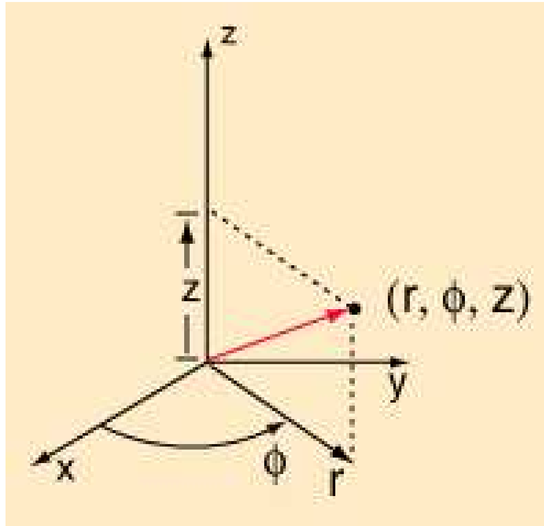
But also $dT = \frac{\partial T}{\partial r}\,dr + \frac{\partial T}{\partial \theta}\,d\theta + \frac{\partial T}{\partial \phi}\,d\phi$

So $a_r = \frac{\partial T}{\partial r}$, $a_\theta = \frac{1}{r}\frac{\partial T}{\partial \theta}$, $a_\phi = \frac{1}{r\sin\theta}\frac{\partial T}{\partial \phi}$

Hence $\boxed{\underline{\nabla}T \;=\; \frac{\partial T}{\partial r}\underline{\hat{r}} + \frac{1}{r}\frac{\partial T}{\partial \theta}\underline{\hat{\theta}} + \frac{1}{r\sin\theta}\frac{\partial T}{\partial \phi}\underline{\hat{\phi}}}$

Cylindrical Polar Coordinates

Unit vectors are $\underline{\hat{r}}$, $\underline{\hat{\theta}}$, (sometimes $\underline{\hat{\phi}}$ is used instead $\underline{\hat{\theta}}$ - diagram) for $\underline{\hat{\theta}}$ and $\underline{k}$ and



$$\begin{aligned} d\underline{r} &= dr\,\underline{\hat{r}} + r\,d\theta\,\underline{\hat{\theta}} + dz\,\underline{k} \\ \text{So}\quad \underline{\nabla}T &= b_r\,\underline{\hat{r}} + b_\theta\,\underline{\hat{\theta}} + b_z\,\underline{k} \end{aligned}$$

$$dT = \underline{\nabla}T \cdot d\underline{r} = b_r\,dr + b_\theta\,d\theta + b_z\,dz$$

But also

$$dT = \frac{\partial T}{\partial r}\,dr + \frac{\partial T}{\partial \theta}\,d\theta + \frac{\partial T}{\partial z}\,dz \text{So}$$

$$b_r = \frac{\partial T}{\partial r},\ b_\theta = \frac{1}{r}\frac{\partial T}{\partial \theta},\ b_z = \frac{\partial T}{\partial z}$$

Hence $\boxed{\underline{\nabla}T = \frac{\partial T}{\partial r}\underline{\hat{r}} + \frac{1}{r}\frac{\partial T}{\partial \theta}\underline{\hat{\theta}} + \frac{\partial T}{\partial z}\underline{k}}$

Example:

$\underline{\nabla}$ Operator

$$
\begin{aligned}
\underline{\nabla} T &= \underline{i}\frac{\partial T}{\partial x} + \underline{j}\frac{\partial T}{\partial y} + \underline{k}\frac{\partial T}{\partial z} \\
&= \left(\underline{i}\frac{\partial}{\partial x} + \underline{j}\frac{\partial}{\partial y} + \underline{k}\frac{\partial}{\partial z}\right) T
\end{aligned}
$$

$$
\boxed{\rightarrow \underline{\nabla} \;=\; \underline{i}\frac{\partial}{\partial x} + \underline{j}\frac{\partial}{\partial y} + \underline{k}\frac{\partial}{\partial z}}
$$

$\underline{\nabla}$ is a vector operator or vector generator. (It operates to the right).
So what is $T\underline{\nabla}$?

Other operations...

Since $\underline{\nabla}$ is a vector operator, can it be used in scalar and vector products with other vectors?

$\underline{\nabla} \cdot \underline{v}$ is called the **divergence** of vector field $\underline{v}$
$\underline{\nabla} \times \underline{v}$ is called the **curl** of vector field $\underline{v}$

Divergence

$$\underline{\nabla} \cdot \underline{v} \quad = \quad \left(\underline{i} \frac{\partial}{\partial x} + \underline{j} \frac{\partial}{\partial y} + \underline{k} \frac{\partial}{\partial z} \right) \cdot \left(\underline{i}\, v_x + \underline{j}\, v_y + \underline{k}\, v_z \right)$$

- A scalar field

Curl

$$\underline{\nabla} \times \underline{v} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix}$$

$$= \underline{i}\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right) + \underline{j}\left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right) + \underline{k}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)$$

- A vector field

Summary

Object	Name	Input	Output
$\underline{\nabla} T$	gradient (grad T)	scalar	vector
$\underline{\nabla} \cdot \underline{v}$	divergence (div $\underline{v}$)	vector	scale
$\underline{\nabla} \times \underline{v}$	curl (curl $\underline{v}$)	vector	vector

Examples: 1) Calculate the divergence and curl of the vector field

$$\underline{F} = \quad xy\underline{i} + yz\underline{j} + xz\underline{k}$$

2) Calculate the divergence of the vector field

$$\underline{v}(x,y) \quad = \quad \underline{\hat{r}} = \frac{x\underline{i} + y\underline{j}}{(x^2+y^2)^{1/2}}$$

3) Calculate the divergence and curl of the vector field

$$\underline{v}(x,y) \quad = \quad \underline{\hat{r}} = \frac{-y\underline{i}}{r} + \frac{x\underline{j}}{r}$$

Second Derivatives

1. $\underline{\nabla} \cdot (\underline{\nabla} T)$ - scalar
2. $\underline{\nabla} \times (\underline{\nabla} T)$ - vector
3. $\underline{\nabla}(\underline{\nabla} \cdot \underline{v})$ - vector
4. $\underline{\nabla} \cdot (\underline{\nabla} \times \underline{v})$ - scalar
5. $\underline{\nabla} \times (\underline{\nabla} \times \underline{v})$ - vector