# Second Order ODEs

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In general second order ODEs contain terms involving $y$, $\frac{dy}{dx}$ , $\frac{d^2y}{dx^2}$ and $F(x)$.
But here only consider equations of the form

$$A\frac{d^2y}{dx^2} + B\frac{dy}{dx} + Cy \quad = \quad 0$$

where $A$, $B$ and $C$ are constants i.e. they are independent of $x$ and $y$. These are known as *homogeneous* equations.
*Solution:* The solution has the form $y = Ke^{\lambda x}$ where $\lambda$ is a constant.
Substitute $y = Ke^{\lambda x}$ in to the ODE to determine the values of $\lambda$.
The equation for $\lambda$ is quadratic so in general there are two values for $\lambda$ which satisfy the equation and so *two* functions $y$ which are solutions to the ODE.

*Demonstration:* Divide equation 1 by $A$ ($A$ is not zero) to get

$$\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy \quad = \quad 0$$

Now substitute $y = Ke^{\lambda x}$

$$K\lambda^2 e^{\lambda x} + bK\lambda e^{\lambda x} + cKe^{\lambda x} \quad = \quad 0$$

Since $e^{\lambda x}$ can never be 0, can divide equation by $Ke^{\lambda x}$ to get

$$\lambda^2 + b\lambda + c \quad = \quad 0$$

This equation is called the *auxiliary equation* of the ODE.
In general there are two values for $\lambda$ which we will call $\lambda_1$ and $\lambda_2$, which can be complex.
The general solution is then

$$y \quad = \quad K_1 e^{\lambda_1 x} + K_2 e^{\lambda_2 x}$$

**BUT**
If the two values of $\lambda$ are real and equal, need to consider a general solution of the form

$$y = (Kx + M)e^{\lambda x}$$

where $K$ and $M$ are constants.

*Example 1:* Find the general solution to the equation

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y \quad = \quad 0$$

*Example 2:* Find the general solution to the equation

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y \quad = \quad 0$$

*Example 3:* Find the equation of motion of a ball of mass $m$ moving along the $x$-axis where the force on the ball is proportional to the distance along the axis.

# Calculus in A Multi-dimensional World

Know about calculus of one variable: If $y = f(x)$, then the slope of the function $f(x)$ at some point $x_1$ is given by the derivative $\frac{df}{dx}$ evaluated at $x_1$. The area $A$ under the curve between $x_1$ and $x_2$ is the integral

$$A = \int_{x_1}^{x_2} f(x)\, dx$$

How do we extend these ideas to functions of several variables?
e.g. temperature in a room (or star): $T(x, y, z)$, pressure in a gas: $P(V, T)$, displacement of the Earth's surface in an earthquake: $Z(x, y, t)$

# Partial Differentiation

Consider a function $f(x, y)$. The partial derivative $\frac{\partial f}{\partial x}$ is defined by

$$\frac{\partial f}{\partial x} = \frac{\lim\limits_{\delta x \to 0} f(x + \delta x, y) - f(x, y)}{\delta x}$$

Often written as $\left(\frac{\partial f}{\partial x}\right)$ or $f_x$. Measures how $f$ changes when $x$ changes.

Evaluated by differentiating $f(x, y)$ w.r.t $x$, **treating y as a constant.** Similarly there is a partial derivative with respect to $y$

$$\frac{\partial f}{\partial y} = \frac{\lim\limits_{\delta y \to 0} f(x, y + \delta y) - f(x, y)}{\delta y}$$

$\dfrac{\partial f}{\partial x}$ is the rate of $f(x,y)$ in the $x$ direction at the point $(x,y)$.
$\dfrac{\partial f}{\partial y}$ is the rate of $f(x,y)$ in the $y$ direction at the point $(x,y)$.
Higher derivatives are also defined and can be computed:

$$\frac{\partial^2 f}{\partial x^2} = f_{xx}, \quad \frac{\partial^2 f}{\partial y^2}, \qquad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = f_{xy}, \quad \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = f_{yx}$$

*Example 4:* Calculate the first and second partial derivatives of $f(x,y) = x^2 - xy + 4y^2$
*Example 5:* Calculate the first partial derivatives of $f(x,y) = \sin(x^2 y)$

# Total Differentials

What is the total change in a function $f(x, y)$ if both $x$ and $y$ change by small amounts $\delta x$ and $\delta y$ ?

$$
\begin{aligned}
\delta f &= f(x+\delta x, y+\delta y) - f(x,y) \\
\delta f &= f(x+\delta x, y+\delta y) - f(x,y) - f(x, y+\delta y) + f(x, y+\delta y) \\
&= \frac{f(x+\delta x, y+\delta y) - f(x, y+\delta y)}{\delta x} + \frac{f(x, y+\delta y) - f(x,y)}{\delta y}\delta y \\
&= \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y \text{ plus higher order powers of } \delta x \text{ and } \delta y
\end{aligned}
$$

So in the limit $\delta x \to 0$ and $\delta y \to 0$, $df = \dfrac{\partial f}{\partial x}dx + \dfrac{\partial f}{\partial y}dy$

$df$ is called the total differential of $f$.

# Application:

1) Suppose $h(x, y)$ is our height above sea level at a position $(x, y)$ and we walk along a path $x(t)$, $y(t)$ where $t$ is the time. How fast do we gain height ?

We want $\dfrac{dh}{dt}$.

Use $dh = \dfrac{\partial h}{\partial x}dx + \dfrac{\partial h}{\partial y}dy$

So $\dfrac{dh}{dt} = \dfrac{\partial h}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial h}{\partial y}\dfrac{dy}{dt}$ which is the generalization of the chain rule.

2) Now suppose we want the rate of change of height with respect to $x$ along the path.

We want $\dfrac{dh}{dx}$.

Since we know the path, we know $y(x)$ and $dy = \dfrac{dy}{dx}dx$ so

$$\begin{aligned} dh &= \frac{\partial h}{\partial x}dx + \frac{\partial h}{\partial y}dy \\ &= \frac{\partial h}{\partial x}dx + \frac{\partial h}{\partial y}\frac{dy}{dx}dx \end{aligned}$$

which gives

$$\frac{dh}{dx} = \frac{\partial h}{\partial x} + \frac{\partial h}{\partial y}\frac{dy}{dx}$$

$\rightarrow$ The total derivative of $h$ w.r.t to $x$ along the path $y(x)$.

# Fields

A field is a quantity which varies with position.

This quantity can be a single number, a scalar, in which case the field is a scalar field e.g. $T(x, y, z)$

Examples: temperature, altitude of land, density

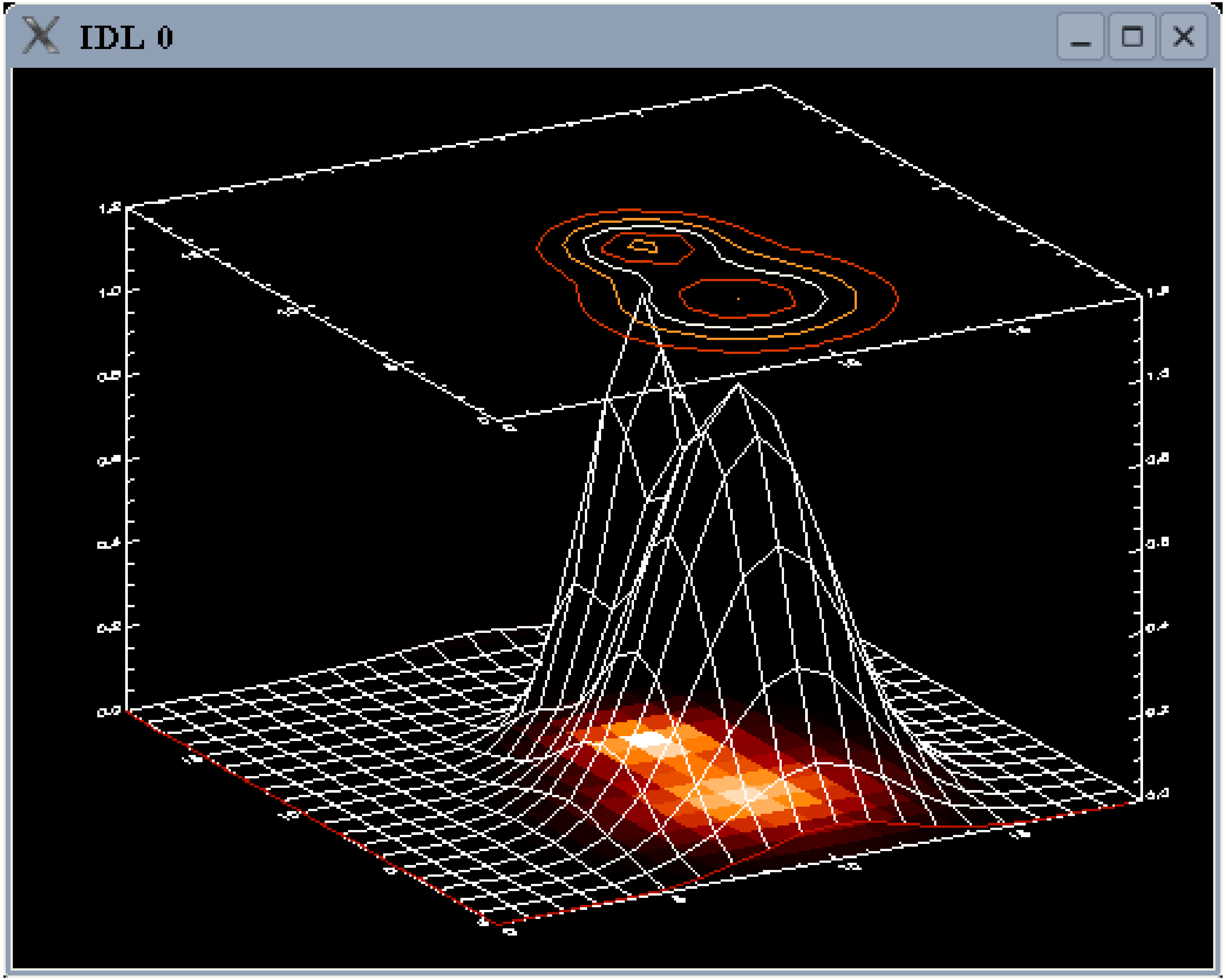
Alternatively the quantity can be a vector, in which case the field is a vector field

Example vector fields: velocity of a fluid, electric field

At each point in space the quantity has both a magnitude and a direction.

*The value of a field at a given point does not depend on the coordinate system.*

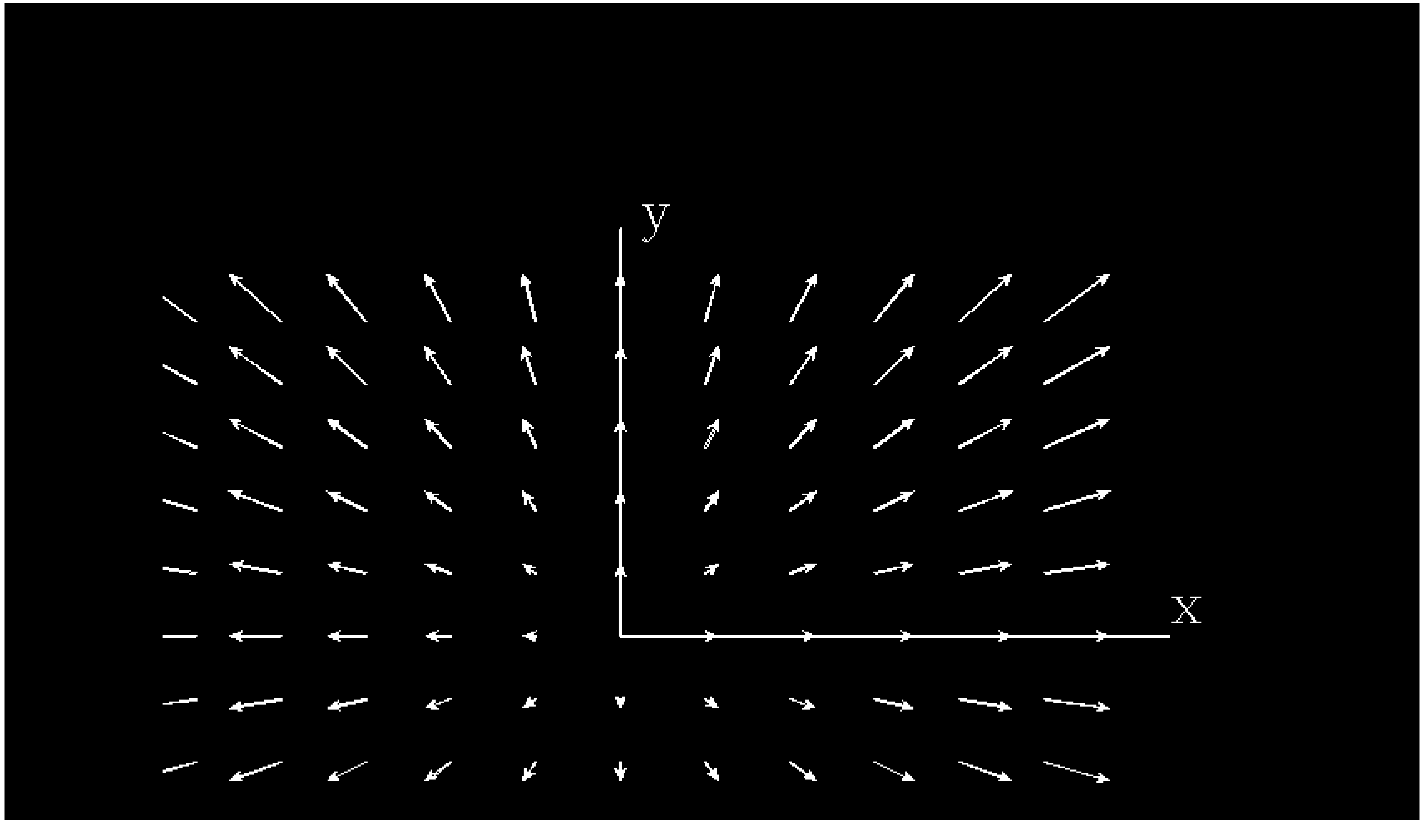
Three different representations of a scalar field:



# Example vector field:

$$\underline{\mathbf{v}}(x,y) \quad = \quad \underline{\hat{\mathbf{r}}} = \frac{x\underline{\mathbf{i}} + y\underline{\mathbf{j}}}{\left(x^2+y^2\right)^{1/2}}$$

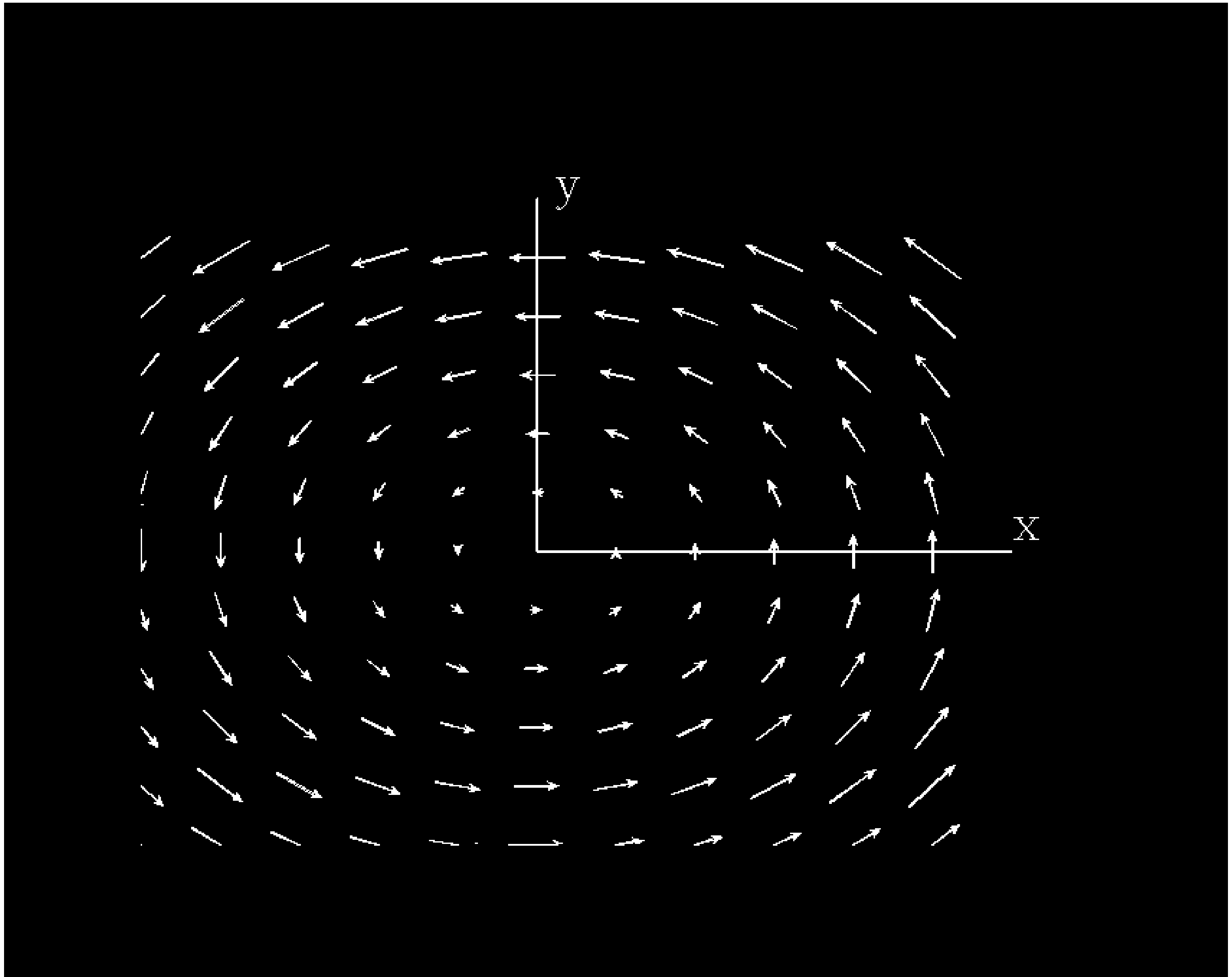
$$\underline{\mathbf{r}} = x\underline{\mathbf{i}} + y\underline{\mathbf{j}} \qquad\qquad r = |\underline{\mathbf{r}}| = \sqrt{x^2+y^2},\ \underline{\hat{\mathbf{r}}} = \frac{\underline{\mathbf{r}}}{r}$$




Diverging vector field

$$\underline{\mathbf{v}}(x,y) \quad = \quad \frac{-y\underline{\mathbf{i}} + x\underline{\mathbf{j}}}{(x^2+y^2)^{3/2}}$$

Curling vector field



# Field Lines

Field lines show the direction of a vector field, but they don't show any information about the magnitude.

## How to Calculate the field lines

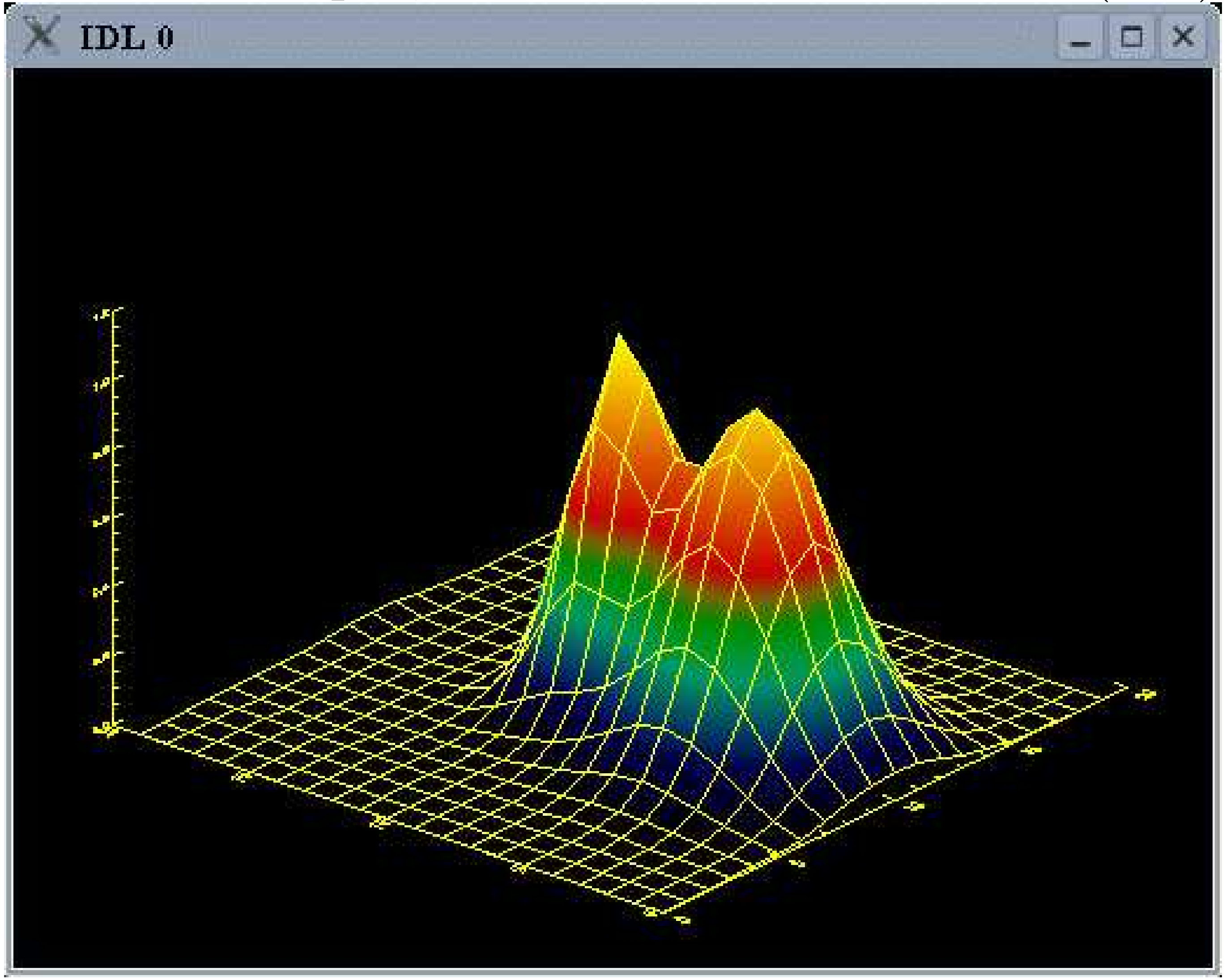
Write a 2-d field as $\underline{\mathbf{v}}(x,y) = v_x(x,y)\underline{\mathbf{i}} + v_y(x,y)\underline{\mathbf{j}}$.
The tanget to the field line makes an angle $\theta$ to the $x$-axis where
$\tan\theta = \dfrac{v_y(x,y)}{v_x(x,y)}$.
Let the field line be $y = f(x)$, but on this field line the slope of the tanget to the field line is $\dfrac{dy}{dx} = \dfrac{v_y(x,y)}{v_x(x,y)}$, which we can integrate to find the equation to the field line.
*Example:* Calculate the field lines of the vector field

$$\underline{\mathbf{v}}(x,y) = \frac{-y\underline{\mathbf{i}} + x\underline{\mathbf{j}}}{(x^2+y^2)^{3/2}}$$

Field lines: $x^2 + y^2 = c$

# The Gradient (Grad)

What's the slope (derivative) of a scalar field $H(x, y)$?



Consider the temperature field $T(x,y,z)$ at the two points $(x,y,z)$ and $(x+dx, y+dy, z+dz)$.
The difference in temperature between these two points is $dT$ where

$$dT = \frac{\partial T}{\partial x}dx + \frac{\partial T}{\partial y}dy + \frac{\partial T}{\partial z}dz$$

But the vector between these two points is $\underline{\mathbf{dr}} = dx\,\underline{\mathbf{i}} + dy\,\underline{\mathbf{j}} + dz\,\underline{\mathbf{k}}$. So we can write

$$\begin{aligned} dT &= \left(\underline{\mathbf{i}}\frac{\partial T}{\partial x} + \underline{\mathbf{j}}\frac{\partial T}{\partial y} + \underline{\mathbf{k}}\frac{\partial T}{\partial z}\right) \cdot \left(dx\,\underline{\mathbf{i}} + dy\,\underline{\mathbf{j}} + dz\,\underline{\mathbf{k}}\right) \\ dT &= \nabla T \cdot \underline{\mathbf{dr}} \end{aligned}$$

where

$$\nabla T = \underline{\mathbf{i}}\frac{\partial T}{\partial x} + \underline{\mathbf{j}}\frac{\partial T}{\partial y} + \underline{\mathbf{k}}\frac{\partial T}{\partial z}$$

which is a **vector field** called the gradient, or grad of the scalar field $T$. Also often written as $grad\ T$.

Expect to see the gradient of scalar fields in various physical laws, e.g. heat flow in 1-D $h = -k\dfrac{\Delta T}{\Delta x}$ while in 3-D $\underline{\mathbf{h}} = -k\nabla T$

Heat flows in the direction of $-$grad $T$, the direction of decreasing $T$ and nrmal to the isotherms.

# Properties of The Gradient

0) $T(x, y, z)$ is a scalar field
1) $\nabla T(x, y, z)$ is a vector field
2) $dT = T(\underline{\mathbf{r}} + \underline{\mathbf{dr}}) - T(r) = \nabla T \cdot \underline{\mathbf{dr}}$ is the change in $T$ between points $\underline{\mathbf{r}}$ and $\underline{\mathbf{r}} + \underline{dr}$
If $\underline{\hat{\mathbf{u}}}$ is a unit vector parallel to $\underline{\mathbf{dr}}$ so $\underline{\mathbf{dr}} = \underline{\hat{u}}\, ds$ ($ds = |\underline{\mathbf{dr}}|$) then
$dT = \nabla T \cdot \underline{\hat{u}}\, ds$ so
3) $\left.\frac{dT}{ds}\right|_{\underline{\hat{u}}} = \nabla T \cdot \underline{\hat{u}}$ is the *directional derivative* - the rate of change of $T$ in the direction of the unit vector $\underline{\hat{u}}$
4) $\nabla T \cdot \underline{\hat{u}}$ is a maximum when $\nabla T$ is parallel to $\underline{\hat{u}}$
5) $|\nabla T|$ is the maximum rate of change and it is in the direction $\nabla T$
Consider the surface $T =$constant. Any point $P$ on this surface $\left.\frac{dT}{ds}\right|_{\underline{\hat{u}}} = 0$ for any vector $u$ tangent to the surface at $P$. i.e. $\nabla T \cdot \left.\frac{dT}{ds}\right|_{\underline{\hat{u}}} = 0$ so
6) $\nabla T$ is normal to the surface of constant $T$.