Example 3: Find and solve the equation of motion of a ball of mass m moving along the x-axis where the force on the ball is proportional to the distance along the axis.

Solution:

Force $F = ma$ where a is the acceleration and $a = \frac{dv}{dt} = \frac{d^2x}{dt^2}$.
But $F = kx$ where k is a constant. So

$$\begin{aligned} m\frac{d^2x}{dt^2} &= kx \\ \frac{d^2x}{dt^2} - \frac{k}{m}x &= 0 \\ \frac{d^2x}{dt^2} - w^2x &= 0 \\ \text{where } w^2 &= \frac{k}{m} \end{aligned}$$

Substitute $x = Ae^{\lambda t}$ giving the auxiliary equation

$$\begin{aligned} \lambda^2 - w^2 &= 0 \\ \text{which has solutions } \lambda &= \pm w \\ \text{If } k > 0, \lambda = \pm w &= \pm\sqrt{\frac{k}{m}} \\ \text{Giving general solution } x &= A_1 e^{t\sqrt{k/m}} + A_2 e^{-t\sqrt{k/m}} \end{aligned}$$

But if $k < 0$, the roots are imaginary, so

$$\begin{aligned} \lambda = \pm w &= \pm\sqrt{\frac{k}{m}} = \pm\sqrt{\frac{-c}{m}} = \pm i\sqrt{\frac{c}{m}} \\ \text{and } c > 0 & \\ \text{Giving general solution } x &= A_1 e^{it\sqrt{c/m}} + A_2 e^{-it\sqrt{c/m}} \end{aligned}$$

where in both cases A_1 and A_2 are constants.

We can expand the exponential terms in terms of $\sin$ and $\cos$ using

$$\begin{aligned} e^{i\sqrt{c/m}} &= \cos(t\sqrt{c/m}) + i\sin(t\sqrt{c/m}) \\ e^{-i\sqrt{c/m}} &= \cos(-t\sqrt{c/m}) + i\sin(-t\sqrt{c/m}) \end{aligned}$$

and so the general solution becomes

$$\begin{aligned} x &= A_1(\cos(t\sqrt{c/m}) + i\sin(t\sqrt{c/m})) + A_2(\cos(-t\sqrt{c/m}) + i\sin(-t\sqrt{c/m})) \\ x &= (A_1 + A_2)\cos(t\sqrt{c/m}) + i(A_1 - A_2)\sin(t\sqrt{c/m}) \end{aligned}$$

So considering either the real or imaginary part, the solution is oscillatory, which is what we would expect as the differential equation in this case, $m\frac{d^2x}{dt^2} = -cx$, is just that of simple harmonic motion.