PC10372, Mathematics 2

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What is this course about?

- Differential Equations
- Vector Calculus

Course Outline

1. Ordinary Differential Equations
   First order (2 lectures)
   Separable, Linear, Homogeneous
   Second Order with constant coefficients (1 lecture)

2. Partial differentiation and the total differential (1 lecture)

3. Scalar fields (2 lectures)
   Gradients, the ∇ operator.

4. Vector fields (2 lectures)
   Discussion of their properties.

5. Differential form of div and curl (2 lectures)
   Combinations of div, grad and curl.
   Concept of divergent and rotational fields.

6. Multiple Integrals (3 lectures)
   Double integrals; triple integrals; spherical polar coordinates.

7. Surface integrals (5 lectures)
   Surface integrals of flux; Conservation laws;
   Evaluation of simple flux integrals,
   Divergence theorem, integral expression for divergence.
8. Line integrals (5 lectures)
   Vector line integrals; conservative forces and potentials;
   Stokes' theorem; integral expression for curl.
9. Irrotational and non-diverging fields (1 lecture)
   Physical Applications

Books

1. Boas, ‘Mathematical Methods in the Physical Sciences’
2. Tinker & Lambourne, ‘Further Mathematics for the Physical Sciences’, Chap. 17 & 18
3. Feynman, ‘Lectures in Physics, Vol. II’, Chapter 2
4. Riley, Hobson & Bence, ‘Mathematical Methods for Physics and Engineering’
5. Schey, ‘Div, grad, curl and all that’

Ordinary Differential Eqns.

What is an ODE?

If $y = f(x)$, then any equation which involves a derivative or derivatives such as $\dfrac{dy}{dx}$ or $\dfrac{d^2y}{dx^2}$ is an ordinary differential equation (ODE).

The *order* of an ODE is highest derivative involved in the equation. For example:

$$\frac{dy}{dx} = x^2 - \text{ first order} \quad \left(\tfrac{dy}{dx}\right)^3 = \cos x - \text{ first order} \quad \frac{d^3y}{dx^3} = x, \text{ third order}$$

First order equations

Need to integrate once and this introduces one constant. For examples:

Solve the first order differential equation $\dfrac{dy}{dx} = 2\cos x$. Integrate both sides with respect to x.

LHS: RHS:

$$\int \frac{dy}{dx}dx \quad = \quad y + A \qquad\qquad \int 2\cos x \; dx \quad = \quad 2\sin x + B$$

where A and B are constants, so we have $y = 2\sin x + c$ where c is an arbitrary constant.

This is the *general solution*. Notice that the general solution to any ODE has as many arbitrary constants as the order of the ODE. So, for example, the general solution of 2nd order ODEs have two constants.

To get a particular solution for a given problem, need the *boundary* (or *initial*) conditions. These provide the means to determine the value of the constant or constants.

So for the above equation, say we want the solution which has $y = 1$ at $x = 0$. Substitute this in to general solution gives $c = 1$. So the particular solution for this case is $y = 2\sin x + 1$.

Setting up ODEs

ODEs are very commen in physiscs.
Example: Write down the ODE describing Newton's law of cooling: the rate of heat loss from a body is proportional to the difference in temperature between a body and its surroundings.
Other examples:
The concentration in a chemical reaction: $\dfrac{dC}{dt} = a - kC$
Forced simple harmonic motion: $\dfrac{d^2x}{dt^2} = F\cos at$
Population growth: $\dfrac{dN}{dt} = \alpha N(N_0 - N)$

1st Order Separable Equations

These have the form $\dfrac{dy}{dx} = f(x)g(y)$

To solve:

1) Rearrange: $\dfrac{1}{g(y)}\dfrac{dy}{dx} = f(x)$

2) Integrate both sides with respect to x:

$$\int \frac{1}{g(y)}\frac{dy}{dx}\ dx \;\; = \;\; \int f(x)\ dx$$

(Note that $\dfrac{dy}{dx}\ dx = dy$ so LHS is $\int \frac{1}{g(y)}\ dy$.)

3) Do both integrals. There will be one constant.

4) Use boundary conditions to evaluate constant.

5) If possible, rearrange to find $y(x)$.

Example: Solve $\dfrac{dy}{dx} = 2xy^2$ given that $y = 1/2$ at $x = 0$.

Example: Solve $x^2(1+y)\dfrac{dy}{dx} = y^2(1-x)$.

Example: Find the general solution of

$$\frac{dT}{dt} = -\alpha(T - T_0)$$

(Newton's Law of cooling). An object has an initial temperature of 100° and after 3 minutes its temperature is 85°. If the air temperature is 40°, find α.

Linear Equations

Linear ODEs have the form

$$\frac{dy}{dx} + P(x)y = Q(x)$$

A linear equation contains no products or functions of y or $\dfrac{dy}{dx}$.So:

$$\begin{aligned}
\left(\frac{dy}{dx}\right)^3 + xy &= 0 \text{ is nonlinear} \\
3x^2\frac{dy}{dx} + (\sin x)y &= \frac{e^x}{1+x} \text{ is linear} \\
y\frac{dy}{dx} + x &= 0 \text{ is nonlinear}
\end{aligned}$$

Linear 1st order ODES are solved using the *integrating factor method*

Integrating factor method

The integrating factor is

$$I(x) \;=\; \exp^{\left(\int P(x)\ dx\right)}$$

Multiply equation by $I(x)$ to get

$$I(x)\frac{dy}{dx} + P(x)I(x)y \;=\; I(x)Q(x)$$

LHS is $\dfrac{d}{dx}(I(x)y)$.
So integrate both sides to get

$$I(x)y \;=\; \int I(x)Q(x)\ dx$$

So

$$y \;=\; e^{-\int P(x)dx}\,(G(x)+c) \text{ where } G = \int e^{\int P(x)\ dx} Q(x)\ dx$$

Example Find the general solution of $\dfrac{dy}{dx} + y\tan x = 3\cos x$

Example Find the solution of $x\dfrac{dy}{dx} + 2y = 4x$ which satisfies $y = 0$ when $x = 1$.

Bernoulli's Equation

$$\frac{dy}{dx} + P(x)\,y \quad = \quad Q(x)\,y^n$$

Not linear ... but can be made linear by a substitution:
1) Rearrange to

$$\frac{1}{y^n}\frac{dy}{dx} + \frac{1}{y^{n-1}}P(x) \quad = \quad Q(x)$$

2) Substitute $u = y^{1-n}$. So

$$\frac{du}{dx} = (1-n)y^{-n}\frac{dy}{dx} \qquad \frac{1}{y^n}\frac{dy}{dx} = \frac{1}{1-n}\frac{du}{dx}$$

3) This gives $\displaystyle\frac{1}{1-n}\frac{du}{dx} + P(x)u = Q(x)$ which is linear and can be solved using an integrating factor.

Example:

Solve $\dfrac{dy}{dx} + \dfrac{y}{x} = 2x^3y^4$.

Integrating Factors: Summary

1. Check form. Rearrange if necessary
2. Find integrating factor: $I = e^{\int p\,dx}$
3. Multiply equnation by I.
4. Write LHS as perfect derivative: $\dfrac{d}{dx}(Iy)$
5. Check by working backwards using product rule !
6. Integrate RHS. Don't forget the constant.
7. Rearrange if possible.
8. Find constant by applying initial conditions.
9. Check solution!

Homogeneous Eqns.

Definition: A function $f(x,y)$ is homogeneous of degree n is

$$f(kx,ky) \quad = \quad k^n f(x,y)$$

This is the case if f is a sum of powers/products of terms in x and y of the same degree n. For example

$$f = x^2 + 3xy + y^2 \qquad \text{Homogeneous degree 2}$$
$$f = 4x^2y^2 + 3y^4 \qquad \text{Homogeneous degree 4}$$

A differential equation of the form $\dfrac{dy}{dx} = \dfrac{f(x,y)}{g(x,y)}$ is *homogeneous* if $f(x,y)$ and $g(x,y)$ are homogeneous of the same degree.

For example:

$$\frac{dy}{dx} = \frac{3x+2y}{y-x}$$

$$\frac{dy}{dx} = \frac{3x+2y}{y^2-x^2}$$

$$\frac{dy}{dx} = \frac{3x^2+2y(x-4)}{y^2}$$

$$\frac{dy}{dx} = \frac{4x^4-x^2y^2+y^4}{3x^4+xy^3+2x^3y}$$

Solutions

Make substitution

$$y = vx \quad \rightarrow \quad \frac{dy}{dx} = v + x\frac{dv}{dx}$$

which will always convert the ODE in to separable form.

Example: Solve $\dfrac{dy}{dx} = \dfrac{x^3 + y^3}{xy^2}$

More generally, any equation of the form $\dfrac{dy}{dx} = F\left(\dfrac{x}{y}\right)$ can be solved using the same substitution. For example

$$\frac{dy}{dx} = \sin\left(\frac{y}{x}\right) \quad v + x\frac{dv}{dx} = \sin v \quad x\frac{dv}{dx} = \sin v - v$$

which is separable.