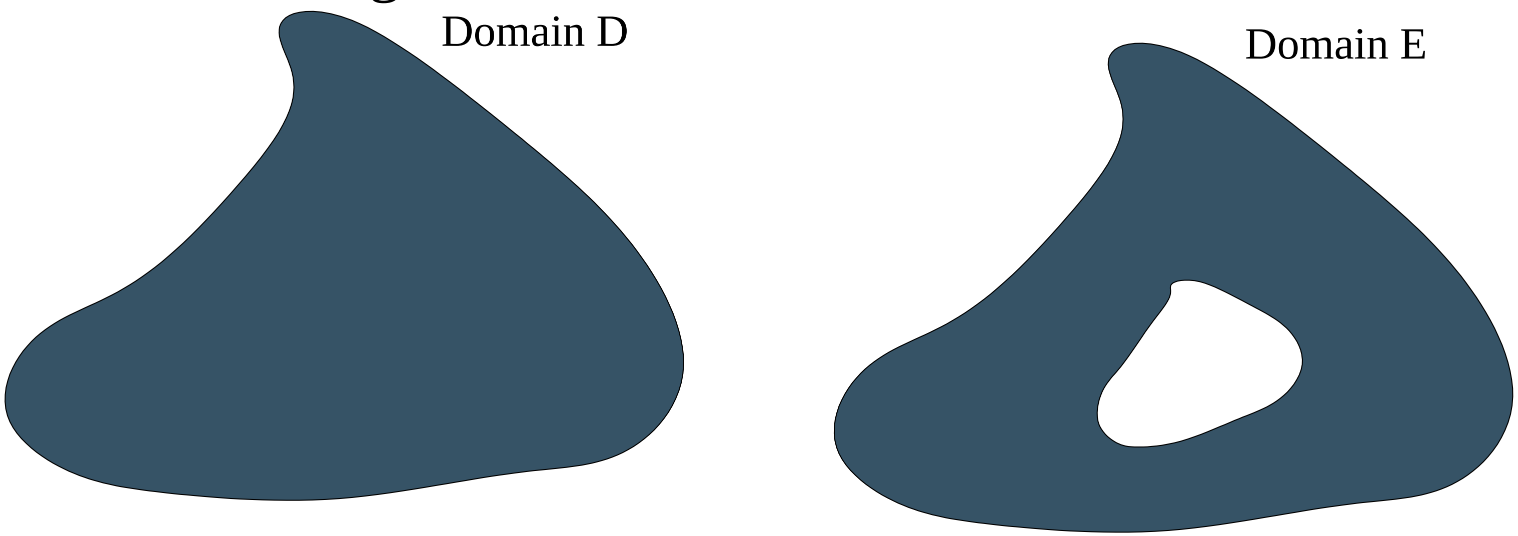
Conservative Fields

In deriving Stoke's theorem implicitly assumed that the vector field was 'smooth' (i.e. continuous, differentiable, and with continous first derivatives) everywhere on the curve C and the surface S.
If D is the domain where the field is smooth, we say that D is a simply connected domain if any close curved C in the domain can be shrunk to a point without leaving D.

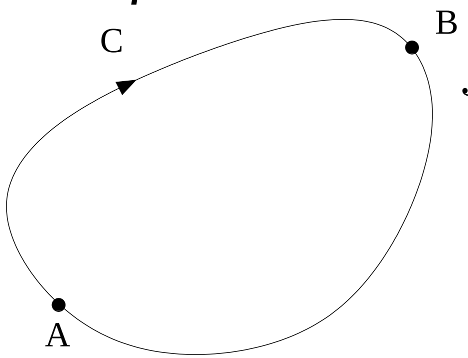


Domain D is simply connected but domain E is not.

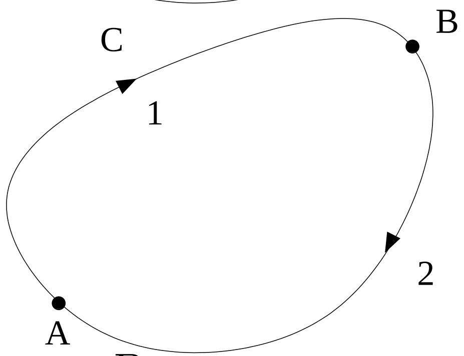
Assume a domain D is simply connected and in this domain there is some vector field $\underline{\mathbf{F}}$ which has $\underline{\nabla} \times \underline{\mathbf{F}} = 0$. Then by Stoke's theorem $\oint \underline{\mathbf{F}} \cdot \underline{\mathbf{d}l} = 0$ for **any** closed path in D.

and $\int_A^B \underline{\mathbf{F}} \cdot \underline{\mathbf{d}l}$ is independent of the path from A to B for any path in the domain D.

Proof: Consider a closed path C which passes through points A and B.



$\oint_C \underline{\mathbf{F}} \cdot \underline{\mathbf{d}l} = 0$. Split the loop into two parts,



$$\int_{A,1}^{B} \underline{\mathbf{F}} \cdot \underline{\mathbf{d}l} \;+\; \int_{B,2}^{A} \underline{\mathbf{F}} \cdot \underline{\mathbf{d}l} = 0$$

$$\int_{A,1}^{B} \underline{\mathbf{F}} \cdot \underline{\mathbf{d}l} \;=\; -\int_{B,2}^{A} \underline{\mathbf{F}} \cdot \underline{\mathbf{d}l}$$

$$\int_{A,1}^{B} \underline{\mathbf{F}} \cdot \underline{\mathbf{d}l} \;=\; \int_{A,2}^{B} \underline{\mathbf{F}} \cdot \underline{\mathbf{d}l}$$

If $\int_A^B \underline{\mathbf{F}} \cdot \underline{\mathbf{d}l}$ is path independent, then $\underline{\mathbf{F}}$ is a *conservative field*

... and $\underline{\mathbf{F}} \cdot \underline{\mathbf{d}l}$ is an *exact differential*. (It's integral depends only on the end points.) So,

$$\begin{aligned}
\underline{\mathbf{F}} \cdot \underline{\mathbf{d}l} &= -dV(x,y,z) = -\frac{\partial V}{\partial x}\,dx - \frac{\partial V}{\partial y}\,dy - \frac{\partial V}{\partial z}\,dz \\
&= -\underline{\nabla} V \cdot \underline{\mathbf{d}l} \\
\text{therefore } \underline{\mathbf{F}} &= -\underline{\nabla} V
\end{aligned}$$

where $V(x,y,z)$ is a scalar field, called the potential, associated with vector field $\underline{\mathbf{F}}$ and V is single valued.
So if $\underline{\nabla} \times \underline{\mathbf{F}} = 0$,
$\int_A^B \underline{\mathbf{F}} \cdot \underline{\mathbf{d}l}$ is path independent
$\underline{\mathbf{F}}$ is conservative (or irrotational)
and there is a scalar potential V such that $\underline{\mathbf{F}} = -\underline{\nabla} V$

Note that if $\underline{\nabla} \cdot \underline{\mathbf{A}} = 0$, $\underline{\mathbf{A}}$ is called solenoidal and $\underline{\mathbf{A}} = \mathrm{curl}\, \underline{\mathbf{B}}$ where $\underline{\mathbf{B}}$ is called the vector potential corresponding to field $\underline{\mathbf{A}}$.

You should now be able to..

1. Solve separable, linear and homogeneous 1st order ODES
2. Solve 2nd order ODEs with constant coefficients
3. Calculate the gradient, $\underline{\nabla}\phi$ and Laplacian, $\underline{\nabla}^2\phi$ of any scalar field ϕ
4. Calculate the divergence, $\underline{\nabla} \cdot \underline{\mathbf{v}}$, and curl, $\underline{\nabla} \times \underline{\mathbf{v}}$, of any vector field $\underline{\mathbf{v}} = \underline{\mathbf{i}}\, v_x(x, y, z) + \underline{\mathbf{j}}\, v_y(x, y, z) + \underline{\mathbf{k}}\, v_z(x, y, z)$
5. Evaluate double and triple integrals, changing the order of integration if required.
6. Evaluate the flux of a vector field through a surface. For a closed surface doing this either directly or applying the divergence theorem
7. Evaluate the line integral of a vector field along a path. For a closed path doing this either directly or by applying Stokes' Theorem.

The Exam...

You do *not* need to remember div, grad, curl or $\underline{\nabla}^2$ in cylindrical or spherical polar coordinates.

- There will be three questions.
- You are required to answer ALL THREE questions

Good luck !