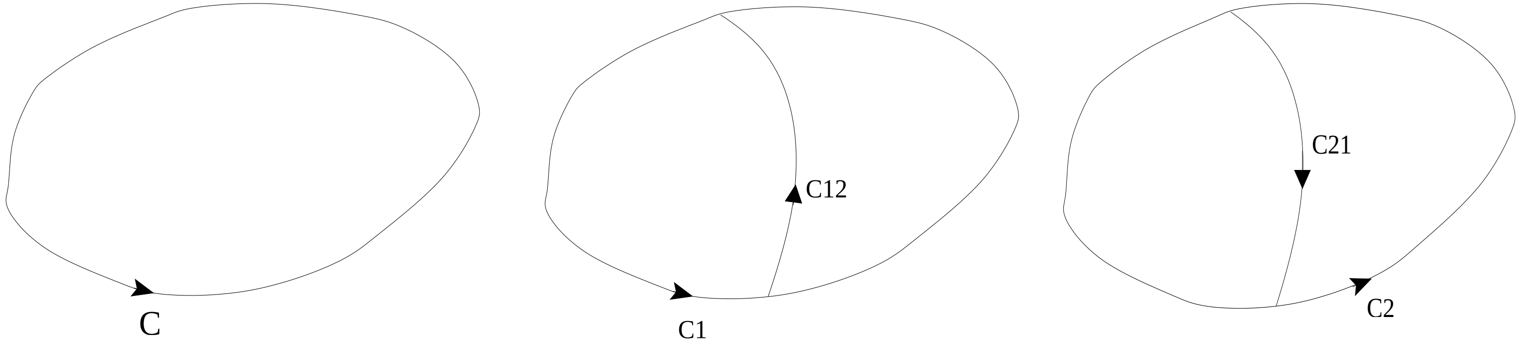
Integral Definition of Curl

$$(\underline{\nabla} \times \underline{\mathbf{v}}) \cdot \underline{\hat{\mathbf{n}}} \quad = \quad \lim_{dA \to 0} \frac{1}{dA} \oint \underline{\mathbf{v}} \cdot \underline{\mathbf{d}l}$$

The component of the curl of a vector field in some direction $\underline{\hat{\mathbf{n}}}$ is equal to the circulation per unit area of the field around a loop to which $\underline{\hat{\mathbf{n}}}$ is the unit normal.

Stokes' Theorem

Divide closed loop C into 2 parts, $C1$ and $C2$.



Consider the circulation of some vector field $\underline{\mathbf{v}}$ around C

$$\begin{aligned}
\oint_C \underline{\mathbf{v}} \cdot \underline{\mathbf{d}l} &= \int_{C1} \underline{\mathbf{v}} \cdot \underline{\mathbf{d}l} + \int_{C2} \underline{\mathbf{v}} \cdot \underline{\mathbf{d}l} \\
&= \int_{C1} \underline{\mathbf{v}} \cdot \underline{\mathbf{d}l} + \int_{C12} \underline{\mathbf{v}} \cdot \underline{\mathbf{d}l} + \int_{C2} \underline{\mathbf{v}} \cdot \underline{\mathbf{d}l} - \int_{C12} \underline{\mathbf{v}} \cdot \underline{\mathbf{d}l} \\
&= \oint_{C1+C12} \underline{\mathbf{v}} \cdot \underline{\mathbf{d}l} + \oint_{C2+C21} \underline{\mathbf{v}} \cdot \underline{\mathbf{d}l}
\end{aligned}$$

In other words the contributions from the common curve in each pair of loops cancel.

Now consider an open surface S bounded by a curve C. Divide the surface in to an infinite number of infinitesimal rectangles. Add up the circulation of $\underline{\mathbf{v}}$ around all these rectangles.The contribution from all the sides common between two rectangles i.e. all the sides *except* those that lay on the curve C, cancel. This leaves only the contribution from the sides on curve C.So

$$\oint_C \underline{\mathbf{v}} \cdot \underline{\mathbf{d}l} \;=\; \sum_i \oint_{\text{loop i}} \underline{\mathbf{v}} \cdot \underline{\mathbf{d}l}$$

But for an infinitesimal rectangular loop

$$\oint_{\text{loop i}} \underline{\mathbf{v}} \cdot \underline{\mathbf{d}l} \;=\; (\underline{\nabla} \times \underline{\mathbf{v}})_i \cdot \underline{\hat{\mathbf{n}}}_i \, dA_i$$

$\underline{\hat{\mathbf{n}}}_i$ normal to surface S for loop i (righthand rule) and dA_i is the area of the loop.

So

$$\oint_C \underline{\mathbf{v}} \cdot \underline{\mathbf{d}l} \;=\; \sum_i (\underline{\nabla} \times \underline{\mathbf{v}})_i \cdot \underline{\hat{\mathbf{n}}}_i \, dA_i$$

In the limit where the loops are infinitesimal, the sum becomes an integral so

$$\oint_C \underline{\mathbf{v}} \cdot \underline{\mathbf{d}l} \;=\; \int_S (\underline{\nabla} \times \underline{\mathbf{v}}) \cdot \underline{\hat{\mathbf{n}}} \, dA$$

Replacing $\underline{\hat{\mathbf{n}}} \, dA$ with $\underline{\mathbf{d}A}$, we get $\boxed{\oint_C \underline{\mathbf{v}} \cdot \underline{\mathbf{d}l} \;=\; \int_S (\underline{\nabla} \times \underline{\mathbf{v}}) \cdot \underline{\mathbf{d}A}}$

- Stokes’ theorem

Example

Maxwell's Equations

Connect one scalar field and three vector fields:

1. ρ = density of electric charge (C/m^3) - scalar field
2. $\underline{\mathbf{E}}$ = electric field strength - vector field
3. $\underline{\mathbf{B}}$ = magnetic field strength - vector field
4. $\underline{\mathbf{J}}$ = current density (A/m^2) - vector field

$$
\begin{array}{ll}
1 & \underline{\nabla} \cdot \underline{\mathbf{E}} = \dfrac{\rho}{\epsilon_0} \\
2 & \underline{\nabla} \cdot \underline{\mathbf{B}} = 0 \\
3 & \underline{\nabla} \times \underline{\mathbf{E}} = -\dfrac{\partial \underline{\mathbf{B}}}{\partial t} \\
4 & \underline{\nabla} \times \underline{\mathbf{B}} = \mu_0 \underline{\mathbf{J}} + \dfrac{1}{c^2}\dfrac{\partial \underline{\mathbf{E}}}{\partial t}
\end{array}
$$

Faraday's Law

Electric field induced by a changing magnetic field of flux Φ of magnetic field $\underline{\mathbf{B}}$:

$$\begin{aligned}
\Phi &= \int_S \underline{\mathbf{B}} \cdot \underline{\mathbf{d}A} \text{ - definition} \\
-\frac{\partial \Phi}{\partial t} &= \oint \underline{\mathbf{E}} \cdot \underline{\mathbf{d}l} \text{ - Faraday's law} \\
-\frac{\partial}{\partial t}\left(\int_S \underline{\mathbf{B}} \cdot \underline{\mathbf{d}A}\right) &= \oint \underline{\mathbf{E}} \cdot \underline{\mathbf{d}l} \\
\oint \underline{\mathbf{E}} \cdot \underline{\mathbf{d}l} &= \int_S (\underline{\nabla} \times \underline{\mathbf{E}}) \cdot \underline{\mathbf{d}A} \\
\int_S (\underline{\nabla} \times \underline{\mathbf{E}}) \cdot \underline{\mathbf{d}A} &= -\frac{\partial}{\partial t}\left(\int_S \underline{\mathbf{B}} \cdot \underline{\mathbf{d}A}\right) = -\int_S \frac{\partial \underline{\mathbf{B}}}{\partial t} \cdot \underline{\mathbf{d}A}
\end{aligned}$$

But this is true for any surface, so

$$\underline{\nabla} \times \underline{\mathbf{E}} = -\frac{\partial \underline{\mathbf{B}}}{\partial t} \text{ - Maxwell 3}$$

Ampère's Law

Magnetic field $\underline{\mathbf{B}}$ due to a current I

$$\oint_C \underline{\mathbf{B}} \cdot \underline{\mathbf{d}l} \;=\; \mu_0 I$$

where the path C encloses the current $I = \int_s \underline{\mathbf{J}} \cdot \underline{\mathbf{d}A}$.
Apply Stokes' theorem

$$\oint_C \underline{\mathbf{B}} \cdot \underline{\mathbf{d}l} \;=\; \int_S (\underline{\nabla} \times \underline{\mathbf{B}}) \cdot \underline{\mathbf{d}A}$$

$$\int_S (\underline{\nabla} \times \underline{\mathbf{B}}) \cdot \underline{\mathbf{d}A} \;=\; \mu_0 \int_S \underline{\mathbf{J}} \cdot \underline{\mathbf{d}A}$$

- must be true for any surface, so

$$\underline{\nabla} \times \underline{\mathbf{B}} \;=\; \mu_0 \underline{\mathbf{J}}$$

This isn't Maxwell 4 - here the current is time independent.

But ...

Can't be true in general. To see why take the divergence of this equation -

$$\begin{aligned} \underline{\nabla} \cdot (\underline{\nabla} \times \underline{\mathbf{B}}) &= 0 \text{ - for any } \underline{\mathbf{B}} \\ \text{so } \underline{\nabla} \cdot \underline{\mathbf{J}} &= 0 \end{aligned}$$

Not in general true. True only for time independent current.
Try

$$\underline{\nabla} \times \underline{\mathbf{B}} = \mu_0 \underline{\mathbf{J}} + \epsilon_0 \mu_0 \frac{\partial \underline{\mathbf{E}}}{\partial t}$$

- take the divergence

$$\begin{aligned} \underline{\nabla} \cdot (\underline{\nabla} \times \underline{\mathbf{B}}) &= \underline{\nabla} \cdot (\mu_0 \underline{\mathbf{J}} + \epsilon_0 \mu_0 \frac{\partial \underline{\mathbf{E}}}{\partial t}) \\ 0 &= \underline{\nabla} \cdot \underline{\mathbf{J}} + \epsilon_0 \frac{\partial}{\partial t} \underline{\nabla} \cdot \underline{\mathbf{E}} \\ \text{but since } \underline{\nabla} \cdot \underline{\mathbf{E}} &= \frac{\rho}{\epsilon_0} \\ \underline{\nabla} \cdot \underline{\mathbf{J}} + \frac{\partial \rho}{\partial t} &= 0 \quad \text{- charge continuity equation} \end{aligned}$$

Maxwell's Equations

$$1 \quad \underline{\nabla} \cdot \underline{\mathbf{E}} = \frac{\rho}{\epsilon_0} \Longleftrightarrow \text{Gauss' law}$$

$$2 \quad \underline{\nabla} \cdot \underline{\mathbf{B}} = 0 \Longleftrightarrow \text{No magnetic charges}$$

$$3 \quad \underline{\nabla} \times \underline{\mathbf{E}} = -\frac{\partial \underline{\mathbf{B}}}{\partial t} \Longleftrightarrow \text{Faraday's Law}$$

$$4 \quad \underline{\nabla} \times \underline{\mathbf{B}} = \mu_0 \underline{\mathbf{J}} + \frac{1}{c^2}\frac{\partial \underline{\mathbf{E}}}{\partial t} \Longleftrightarrow \text{Ampère's Law}$$

EM Waves

Can use Maxwell equations to derive wave equation for em waves.Assume free space and no current.

$$
\begin{aligned}
\underline{\nabla} \times \underline{\mathbf{E}} &= -\frac{\partial \underline{\mathbf{B}}}{\partial t} \\
\underline{\nabla} \times \underline{\mathbf{B}} &= \epsilon_0 \mu_0 \frac{\partial \underline{\mathbf{E}}}{\partial t}
\end{aligned}
$$

Take curl

$$
\begin{aligned}
\underline{\nabla} \times (\underline{\nabla} \times \underline{\mathbf{B}}) &= \epsilon_0 \mu_0 \frac{\partial \underline{\nabla} \times \underline{\mathbf{E}}}{\partial t} = -\epsilon_0 \mu_0 \frac{\partial^2 \underline{\mathbf{B}}}{\partial t^2} \\
\underline{\nabla}(\underline{\nabla} \cdot \underline{\mathbf{B}}) - \underline{\nabla}^2 \underline{\mathbf{B}} &= -\epsilon_0 \mu_0 \frac{\partial^2 \underline{\mathbf{B}}}{\partial t^2} \\
\underline{\nabla}^2 \underline{\mathbf{B}} &= \frac{1}{c^2} \frac{\partial^2 \underline{\mathbf{B}}}{\partial t^2}
\end{aligned}
$$

where $c = 1/\sqrt{\epsilon_0 \mu_0}$ - The wave equation for EM waves.