PC10372, Mathematics 2
Example Sheet 9 Solutions

1) Use spherical polar coordinates. For the curved surface for which the radius is fixed with value R and $x = R\sin\theta\cos\phi$,

$$\begin{aligned}
\underline{\mathbf{d}}A &= R^2 \sin\theta \, d\theta \, d\phi \, \hat{\underline{\mathbf{r}}} \\
\underline{\mathbf{v}} &= R^2 \sin^2\theta \cos^2\phi \, \underline{\mathbf{k}} \\
\text{So, } \underline{\mathbf{v}} \cdot \underline{\mathbf{d}}A &= R^4 \sin^3\theta \cos^2\phi \, \hat{\underline{\mathbf{r}}} \cdot \underline{\mathbf{k}} \, d\theta \, d\phi \\
\text{noting that } \hat{\underline{\mathbf{r}}} \cdot \underline{\mathbf{k}} &= \cos\theta \\
\int_{\text{curve}} \underline{\mathbf{v}} \cdot \underline{\mathbf{d}}A &= R^4 \int_0^{2\pi} d\phi \, \cos^2\phi \int_0^{\pi/2} d\theta \, \sin^3\theta \cos\theta \\
\text{using } \int_0^{2\pi} d\phi \, \cos^2\phi &= \frac{1}{2}\int_0^{2\pi} d\phi (1+\cos 2\theta) = \frac{1}{2}[\phi + \frac{1}{2}\sin 2\theta]_0^{2\pi} = \pi \\
\text{and } \int_0^{\pi/2} d\theta \, \sin^3\theta\cos\theta &= \frac{1}{4}[\sin^4\theta]_0^{\pi/2} = \frac{1}{4} \\
\text{gives } \int_{\text{curve}} \underline{\mathbf{v}} \cdot \underline{\mathbf{d}}A &= R^4 \pi \frac{1}{4}
\end{aligned}$$

Now for the flat surface. But notice here that the radius is variable, $0 \le r \le R$, but of course θ is a constant and equal to $\pi/2$ and $x = r\cos\theta$

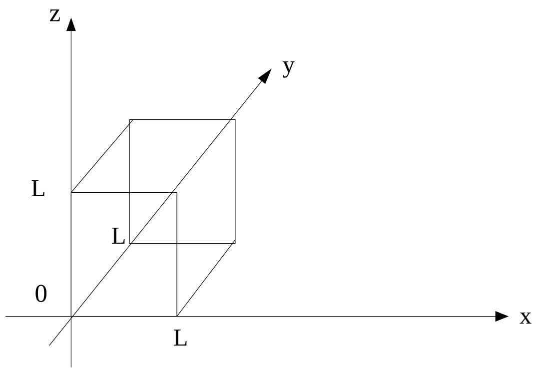
$$\begin{aligned}
\underline{\mathbf{d}}A &= -r \, dr \, d\phi \, \underline{\mathbf{k}} \\
\underline{\mathbf{v}} &= r^2 \cos^2\phi \, \underline{\mathbf{k}} \\
\text{So, } \underline{\mathbf{v}} \cdot \underline{\mathbf{d}}A &= -r^3 \cos^2\phi \, dr \, d\phi \\
\int_{\text{flat}} \underline{\mathbf{v}} \cdot \underline{\mathbf{d}}A &= -\int_0^{2\pi} \cos^2\phi \, d\phi \int_0^R r^3 \, dr = -\pi\frac{R^4}{4}
\end{aligned}$$

Adding these to get the total flux gives

$$\int_S \underline{\mathbf{v}} \cdot \underline{\mathbf{d}}A = 0$$

This result could have been obtained in one line using the divergence theorem as $\nabla \cdot \underline{\mathbf{v}} = 0$.

2) Calculate the flux through each face of the box:



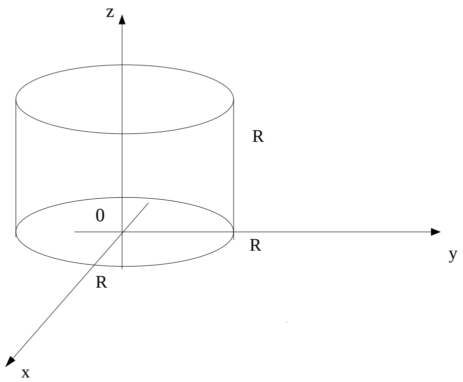
left face: $x = 0 \; \underline{\mathbf{d}A} = -dy\,dz\,\underline{\mathbf{i}} \rightarrow \underline{\mathbf{r}} \cdot \underline{\mathbf{d}A} = -x\,dy\,dz|_{x=0} = 0$
front face: $y = 0 \; \underline{\mathbf{d}A} = -dx\,dz\,\underline{\mathbf{j}} \rightarrow \underline{\mathbf{r}} \cdot \underline{\mathbf{d}A} = -y\,dx\,dz|_{y=0} = 0$
bottom face: $z = 0 \; \underline{\mathbf{d}A} = -dx\,dy\,\underline{\mathbf{k}} \rightarrow \underline{\mathbf{r}} \cdot \underline{\mathbf{d}A} = -z\,dx\,dy|_{z=0} = 0$

right face: $x = L \; \underline{\mathbf{d}A} = dy\,dz\,\underline{\mathbf{i}} \rightarrow \underline{\mathbf{r}} \cdot \underline{\mathbf{d}A} = x\,dy\,dz|_{x=L} = L\,dy\,dz$
back face: $y = L \; \underline{\mathbf{d}A} = dx\,dz\,\underline{\mathbf{j}} \rightarrow \underline{\mathbf{r}} \cdot \underline{\mathbf{d}A} = y\,dx\,dz|_{y=L} = L\,dx\,dz$
top face: $z = L \; \underline{\mathbf{d}A} = dx\,dy\,\underline{\mathbf{k}} \rightarrow \underline{\mathbf{r}} \cdot \underline{\mathbf{d}A} = z\,dy\,dx|_{x=L} = L\,dy\,dx$

Giving a total flux out of the cube

$$\int_S \underline{\mathbf{r}} \cdot \underline{\mathbf{d}A} \;\; = \;\; L \int_{\text{right}} dy\,dz + L \int_{\text{back}} dx\,dz + L \int_{\text{top}} dx\,dy = L^3 + L^3 + L^3 = 3L^3$$

3) Use cylindrical polar coordinates



Consider curved and the two flat surfaces separately.
Bottom surface: The vector $\underline{\mathbf{r}}$ lies in this surface and so the normal to the surface, $\underline{\hat{\mathbf{n}}} = -\underline{\mathbf{k}}$,

is perpendicular to this and so $\underline{\mathbf{r}} \cdot \underline{\mathbf{d}A} = 0$.

Top Surface: $\underline{\mathbf{d}A} = r\,dr\,d\theta\underline{\mathbf{k}}, z = R, 0 \leq r \leq R, 0 \leq \theta \leq 2\pi$ so

$$\begin{aligned} \underline{\mathbf{r}} \cdot \underline{\mathbf{d}A} &= zr\,dr\,d\theta|_{z=R} = Rr\,dr\,d\theta \\ \int_{\text{top}} \underline{\mathbf{r}} \cdot \underline{\mathbf{d}A} &= R\int_0^R r\,dr \int_0^{2\pi} d\theta = \pi R^3 \end{aligned}$$

Curved Surface: For the field $\underline{\mathbf{r}} = x\underline{\mathbf{i}} + y\underline{\mathbf{j}} + z\underline{\mathbf{k}}$, the $\underline{\mathbf{k}}$ component is parallel to the curved surface i.e. perpendicular to the normal to the curved surface, and so does not contribute to the flux. The remaining part $x\underline{\mathbf{i}} + y\underline{\mathbf{j}}$ is normal to the surface, so writing $\underline{\mathbf{d}A} = \underline{\hat{\mathbf{r}}}\,dA$. The length of the radius vector is $(x^2 + y^2)^{1/2} = R$ on the surface, so $\underline{\mathbf{r}} \cdot \underline{\hat{\mathbf{r}}} = R$ and

$$\int_{\text{curved}} \underline{\mathbf{r}} \cdot \underline{\mathbf{d}A} = R\int_{\text{curved}} dA = R2\pi RR = 2\pi R^3$$

where $2\pi R^2$ is the area of the curved surface. Therefore the total flux out of the cylinder is $\pi R^3 + 2\pi R^3 = 3\pi R^3$

4) For a sphere in spherical polar coordinates, $\underline{\mathbf{d}A} = R^2 \sin\theta\,d\theta\,d\phi\underline{\hat{\mathbf{r}}}$, so

$$\begin{aligned} \underline{\mathbf{r}} \cdot \underline{\mathbf{d}A} &= r^2 \sin\theta\,d\theta\,d\phi\,\underline{\mathbf{r}} \cdot \underline{\hat{\mathbf{r}}}|_{r=R} \\ &= R^3 \sin\theta\,d\theta\,d\phi \text{ where we have used } \underline{\mathbf{r}} \cdot \underline{\hat{\mathbf{r}}} = R \text{ on the surface of the sphere.} \\ \int_S \underline{\mathbf{r}} \cdot \underline{\mathbf{d}A} &= R^3 \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta\,d\theta \\ &= R^3\,2\pi\,[-\cos\theta]_0^{\pi} = 2 \\ \text{Flux} &= 4\pi\,R^3 \end{aligned}$$

So looking at the results of parts 2, 3 and 4 we can see that in each case the flux is equal to three times the volume of the region, so

$$\text{Volume of shape} = \frac{1}{3}\int_S \underline{\mathbf{r}} \cdot \underline{\mathbf{d}A}$$