PC10372, Mathematics 2
Example Sheet 8 Solutions

1)

$$\text{a) } \int_0^1 dy \int_1^2 dx\, xy^2 = \int_0^1 dy\, y^2 \int_1^2 dx\, x = \left[\frac{y^3}{3}\right]_0^1 \left[\frac{x^2}{2}\right]_1^2 = \frac{1}{3}(2-\frac{1}{2}) = \frac{1}{2}$$

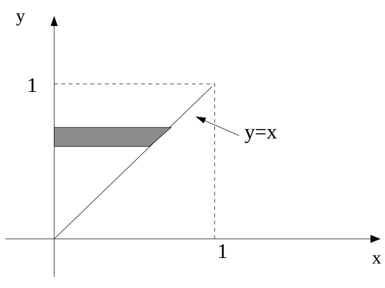
$$\begin{aligned}
\text{b) } \int_0^{\pi/2} dx \int_0^{\pi/2} dy\ \cos(x+y) &= \int_0^{\pi/2} dx\ [\sin(x+y)]_{y=0}^{y=\pi/2} \\
&= \int_0^{\pi/2} dx\,(\sin(x+\pi/2) - \sin x) = \int_0^{\pi/2} dx\,(\cos x - \sin x) \\
&= [\sin x + \cos x]_0^{\pi/2} = 1 + 0 - 0 - 1 = 0
\end{aligned}$$

$$\text{c) } \int_0^1 dy \int_0^1 dx\, \frac{x}{y} = \int_1^2 \frac{dy}{y} \int_0^1 dx\, x = [\ln y]_1^2 [\frac{x^2}{2}]_0^1 = \frac{1}{2}\ln 2$$

2)

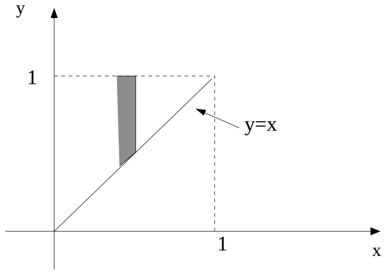
$$\begin{aligned}
V &= \int\!\!\int_R\!\!\int dx\, dy\, dz = \int dx \int dy \int_0^{z=2x-y+3} \\
&= \int_0^1 dx \int_0^2 dy (2x - y + 3) = \int_0^1 \left(2x[y]_0^2 - \left[\frac{y^2}{2}\right]_0^2 + 3[y]_0^2\right) \\
&= \int_0^1 dx(4x - 2 + 6) = \int_0^1 dx(4x+4) = [2x^2 + 4x]_0^1 = 6
\end{aligned}$$

3)



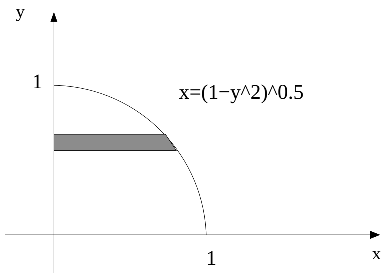
$$\text{a) } A = \int_0^1 dy \int_0^y dx = \int_0^1 dy\, y = \frac{1}{2}[y^2]_0^1 = \frac{1}{2}$$

Reverse the order of integration



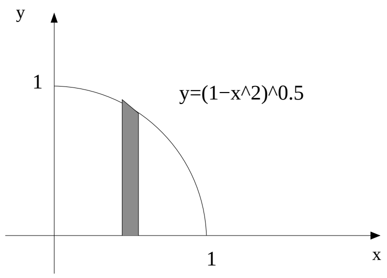
$$\begin{aligned} A &= \int_0^1 dx \int_x^1 dy = \int_0^1 dx [y]_x^1 = \int_0^1 dx\,(1-x) \\ &= \left[x - \frac{x^2}{2}\right]_0^1 = 1 - \frac{1}{2} = \frac{1}{2} \end{aligned}$$

b)



$$\begin{aligned} A &= \int_0^1 dy \int_0^{\sqrt{1-y^2}} dx = \int_0^1 dy [x]_0^{\sqrt{1-y^2}} = \int_0^1 dy \sqrt{1-y^2} \\ \text{Substitute} &\quad y = \sin\theta,\ dy = \cos\theta\, d\theta \\ \int_0^{\pi/2} d\theta\ \cos^2\theta &= \frac{1}{2}\int_0^{\pi/2} d\theta\,(1+\cos 2\theta) = \frac{1}{2}\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\pi/2} = \frac{\pi}{4} \end{aligned}$$

b)



$A = \int_0^1 dx \int_0^{\sqrt{1-x^2}} dy$ which is clearly the same integral but with the x and y swapped.

4)

a) $I = \int_R xyz\, dx\, dy\, dz = \int_0^2 dx\, x \int_0^4 dy\, y \int_2^4 dz\, z = \left[\frac{x^2}{2}\right]\left[\frac{y^2}{2}\right]\left[\frac{z^2}{2}\right]_2^4 = 2 \times 2 \times (8-2) = 24$

b) Since the the volume of integration is a sphere, with $r \leq 2$, use spherical polar coordinates

$$\begin{aligned} dx\,dy\,dz &= r^2 \sin\theta\,dr\,d\theta\,d\phi \\ r = &\cos\theta, x^2 + y^2 + z^2 = r^2 \end{aligned}$$

So

$$\begin{aligned} I &= \int_R z^4\,dx\,dy\,dz = \int_0^2 dr \int_0^{\pi} d\theta \int_0^{2\pi} d\phi\, r^2 \sin\theta\,(\cos\theta)^4 = \int_0^2 r^6\,dr \int_0^{\pi} d\theta \sin\theta \cos^4\theta \int_0^{2\pi} d\phi \\ &= \left[\frac{r^7}{7}\right] \left[-\frac{1}{5}\cos^5\theta\right]_0^{\pi} [\phi]_0^{2\pi} = \frac{128}{7}\left(\frac{1}{5} + \frac{1}{5}\right) 2\pi = \frac{512}{35}\pi \end{aligned}$$