PC10372, Mathematics 2
Example Sheet 7
Solutions

1) $f(r)\underline{\mathbf{r}}$ is a radially directed field with the field lines pointing outwards if $f(r) > 0$ or inwards (towards the origin) if $f(r) < 0$.

Use the identity $\nabla \times (\phi\underline{\mathbf{v}}) = \phi\nabla \times \underline{\mathbf{v}} - \underline{\mathbf{v}} \times \nabla\phi$ with $\phi = f(r)$ and $\underline{\mathbf{v}} = \underline{\mathbf{r}}$. Then

$$\begin{aligned}\nabla \times (f(r)\underline{\mathbf{r}}) &= f(r)\nabla \times \underline{\mathbf{r}} - \underline{\mathbf{r}} \times \nabla f(r) \\ &= -\underline{\mathbf{r}} \times \nabla f(\underline{\mathbf{r}})\end{aligned}$$

since $\nabla \times \underline{\mathbf{r}} = 0$.

$\nabla f(r) = \frac{df}{dr}\hat{\underline{\mathbf{r}}}$, so it is parallel $\underline{\mathbf{r}}$, pointing in the radial direction. Remember that the cross product of two parallel vectors is zero so

$$\begin{aligned}\underline{\mathbf{r}} \times \nabla f(r) &= 0, \quad \text{and so} \\ \nabla \times (f(r)\underline{\mathbf{r}}) &= 0\end{aligned}$$

2)

$$g(r)\underline{\mathbf{k}} \times \underline{\mathbf{r}} = g(r)\left\|\begin{array}{ccc}\underline{\mathbf{i}} & \underline{\mathbf{j}} & \underline{\mathbf{k}} \\ 0 & 0 & 1 \\ x & y & z\end{array}\right\| = g(r)\left(-y\underline{\mathbf{i}} + x\underline{\mathbf{j}}\right)$$

The field lines are given by

$$\begin{aligned}\frac{dy}{dx} &= \frac{g(r)x}{-g(r)y} = \frac{-y}{x} \\ \int y\,dy &= \int x\,dx \\ y^2 + x^2 &= \text{constant}\end{aligned}$$

So the field lines are circles.

Let $g(r)\underline{\mathbf{k}} = \underline{\mathbf{v}}$ and $\underline{\mathbf{r}} = \underline{\mathbf{w}}$ and use

$$\begin{aligned}\nabla \cdot (\underline{\mathbf{v}} \times \underline{\mathbf{w}}) &= (\nabla \times \underline{\mathbf{v}}) \cdot \underline{\mathbf{w}} - (\nabla \times \underline{\mathbf{w}}) \cdot \underline{\mathbf{v}} \\ \rightarrow \nabla \cdot (g(r)\underline{\mathbf{k}} \times \underline{\mathbf{r}}) &= (\nabla \times g(r)\underline{\mathbf{k}}) \cdot \underline{\mathbf{r}} - (\nabla \times \underline{\mathbf{r}}) \cdot \underline{\mathbf{k}}\, g(r)\end{aligned}$$

but the last term is zero as $\nabla \times \underline{\mathbf{r}} = 0$. Now use

$$\begin{aligned} \nabla \cdot (\phi \underline{\mathbf{v}}) &= \phi \nabla \cdot \underline{\mathbf{v}} + \underline{\mathbf{v}} \cdot \nabla \phi \\ \rightarrow \nabla \cdot (g(r) \underline{\mathbf{k}} \times \underline{\mathbf{r}}) &= -(\underline{\mathbf{k}} \times \nabla g(r)) \cdot \underline{\mathbf{r}} \\ &= \underline{\mathbf{r}} \cdot (\underline{\mathbf{k}} \times \nabla g(r)) \end{aligned}$$

- as required. Now $\nabla g(r)$ is parallel to $\underline{\mathbf{r}}$, so $\nabla g(r) \times \underline{\mathbf{k}}$ is normal to $\underline{\mathbf{r}}$, so $\underline{\mathbf{r}} \cdot (\underline{\mathbf{k}} \times \nabla g(r)) = 0$, so

$$\nabla \cdot (g(r) \underline{\mathbf{k}} \times \underline{\mathbf{r}}) = 0$$

3) Using Cartesian coordinates
a) $\nabla^2 (x^3 - 3xy^2 + y^3) = 6x - 6x + 6y = 6y$
b)

$$\begin{aligned} \nabla^2 \ln(x^2 + y^2) &= \frac{\partial}{\partial x}\left(\frac{2x}{x^2+y^2}\right) + \frac{\partial}{\partial y}\left(\frac{2y}{x^2+y^2}\right) \\ &= \frac{2}{x^2+y^2} - \frac{4x^2}{(x^2+y^2)^2} + \frac{2}{x^2+y^2} - \frac{4y^2}{(x^2+y^2)^2} \\ &= \frac{4}{x^2+y^2} - \frac{4(x^2+y^2)}{(x^2+y^2)^2} = 0 \end{aligned}$$

In cylindrical polars:

$$\nabla^2 \ln r^2 = 2\nabla^2 \ln r = \frac{2}{r}\frac{\partial}{\partial r}\left(r \frac{\partial}{\partial r}(\ln r)\right) = \frac{2}{r}\frac{\partial}{\partial r} 1 = 0$$

c)

$$\begin{aligned} \nabla^2 (x^2 - y^2)^{1/2} &= \frac{\partial}{\partial x}\left(\frac{x}{(x^2-y^2)^{1/2}}\right) + \frac{\partial}{\partial y}\left(\frac{-y}{(x^2-y^2)^{1/2}}\right) \\ &= \frac{1}{(x^2-y^2)^{1/2}} - \frac{x^2}{(x^2-y^2)^{3/2}} - \frac{1}{(x^2-y^2)^{1/2}} - \frac{y^2}{(x^2-y^2)^{3/2}} \\ &= -\frac{x^2+y^2}{(x^2-y^2)^{3/2}} \end{aligned}$$

d)

$$\nabla^2 \left(x^2 + y^2 + z^2\right)^{-1/2} = \frac{\partial}{\partial x}\left(\frac{-x}{(x^2+y^2+z^2)^{3/2}}\right) + \frac{\partial}{\partial y}\left(\frac{-y}{(x^2+y^2+z^2)^{3/2}}\right)$$

$$
\begin{aligned}
&+\frac{\partial}{\partial z}\left(\frac{-z}{(x^2+y^2+z^2)^{3/2}}\right) \\
&= -\frac{1}{r^3}+\frac{3x^2}{r^5}-\frac{1}{r^3}+\frac{3y^2}{r^5}-\frac{1}{r^3}+\frac{3z^2}{r^5}=-\frac{3}{r^3}+\frac{3(x^2+y^2+z^2)}{r^5} \\
&= 0
\end{aligned}
$$

In spherical polars:

$$
\nabla^2\left(\frac{1}{r}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\frac{1}{r}\right)=\frac{2}{r^2}\frac{\partial}{\partial r}-1=0
$$

e)

$$
\begin{aligned}
\nabla^2\left(\ln\left(x^2+y^2+z^2\right)^{-1/2}\right) &= \frac{\partial}{\partial x}\left(\frac{2x}{x^2+y^2+z^2}\right)+\frac{\partial}{\partial y}\left(\frac{2y}{x^2+y^2+z^2}\right) \\
&\quad +\frac{\partial}{\partial z}\left(\frac{2z}{x^2+y^2+z^2}\right) \\
&= \frac{2}{r^2}-\frac{4x^2}{r^4}+\frac{2}{r^2}-\frac{4y^2}{r^4}+\frac{2}{r^2}-\frac{4z^2}{r^4}=\frac{6}{r^2}-4\frac{x^2+y^2+z^2}{r^4} \\
&= \frac{2}{r^2}
\end{aligned}
$$

In spherical polars:

$$
\nabla^2\ln r^2 = 2\nabla^2\ln r=\frac{2}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}(\ln r)\right)=\frac{2}{r^2}\frac{\partial}{\partial r}r=\frac{2}{r^2}
$$

Notice that b), d) and e) are all much simpler in polar coordinates.