PC10372, Mathematics 2
Example Sheet 6 Solutions

1)

$$\begin{aligned}
\underline{\mathbf{v}} &= \left(x^2z\underline{\mathbf{i}} + y^2x\underline{\mathbf{j}} + z^2y\underline{\mathbf{k}}\right)/2 \\
\nabla\cdot\underline{\mathbf{v}} &= (xz + yx + zy) \\
\nabla(\nabla\cdot\underline{\mathbf{v}}) &= (z+y)\underline{\mathbf{i}} + (x+z)\underline{\mathbf{j}} + (x+y)\underline{\mathbf{k}} \\
\nabla^2\underline{\mathbf{v}} &= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\underline{\mathbf{v}} = z\underline{\mathbf{i}} + x\underline{\mathbf{j}} + y\underline{\mathbf{k}} \\
\rightarrow \nabla(\nabla\cdot\underline{\mathbf{v}}) - \nabla^2\underline{\mathbf{v}} &= y\underline{\mathbf{i}} + z\underline{\mathbf{j}} + x\underline{\mathbf{k}} \\
\nabla\times\underline{\mathbf{v}} &= \begin{vmatrix} \underline{\mathbf{i}} & \underline{\mathbf{j}} & \underline{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2z/2 & y^2x/2 & z^2y/2 \end{vmatrix} = \frac{z^2}{2}\underline{\mathbf{i}} + \frac{x^2}{2}\underline{\mathbf{j}} + \frac{y^2}{2}\underline{\mathbf{k}} \\
\nabla\times(\nabla\times\underline{\mathbf{v}}) &= \begin{vmatrix} \underline{\mathbf{i}} & \underline{\mathbf{j}} & \underline{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z^2/2 & x^2/2 & y^2/2 \end{vmatrix} = y\underline{\mathbf{i}} + z\underline{\mathbf{j}} + x\underline{\mathbf{k}} \\
&= \nabla(\nabla\cdot\underline{\mathbf{v}}) - \nabla^2\underline{\mathbf{v}} \text{ Q.E.D.}
\end{aligned}$$

2)

$$\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t} \qquad \nabla\times\mathbf{B} = \frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t}$$

$$\begin{aligned}
\nabla\times(\nabla\times\mathbf{E}) = -\frac{\partial}{\partial t}(\nabla\times\mathbf{B}) &= -\frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2} \\
\rightarrow \nabla(\nabla\cdot\mathbf{E}) - \nabla^2\mathbf{E} &= -\frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2} \\
\text{But } \nabla\cdot\mathbf{E} = 0, \text{ so,} & \\
\nabla^2\mathbf{E} &= \frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2}
\end{aligned}$$

which is the wave equation.

Similarly,

$$\nabla\times(\nabla\times\mathbf{B}) = \frac{1}{c^2}\frac{\partial}{\partial t}(\nabla\times\mathbf{E}) = -\frac{1}{c^2}\frac{\partial^2\mathbf{B}}{\partial t^2}$$

$$\rightarrow \nabla(\nabla \cdot \mathbf{B}) - \nabla^2 \mathbf{B} \ = \ -\frac{1}{c^2}\frac{\partial^2 \mathbf{B}}{\partial t^2}$$

$$\text{But } \nabla \cdot \mathbf{B} = 0, \qquad \text{so } \nabla^2 \mathbf{B} = \frac{1}{c^2}\frac{\partial^2 \mathbf{B}}{\partial t^2}$$

again a wave equation.

Substituting $\mathbf{E} = \underline{\mathbf{i}} \sin(kz - \omega t)$ into the wave equation for $\mathbf{E}$

$$\nabla^2 \mathbf{E} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\mathbf{E} \ = \ \frac{\partial^2 \mathbf{E}}{\partial z^2} = -k^2 \mathbf{E}$$

$$\text{and } \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} \ = \ -\frac{\omega^2}{c^2}\mathbf{E}$$

So $\mathbf{E} = \underline{\mathbf{i}} \sin(kz - \omega t)$ is a solution to the wave equation if $\omega = ck$ giving $\mathbf{E} = \underline{\mathbf{i}} \sin k(z - ct)$. This is a plane wave travelling in the z-direction at speed c with the electric field in the x-direction.

3) Using the relation discussed lecture

$$\nabla \cdot (\underline{\mathbf{v}} \times \underline{\mathbf{w}}) = (\nabla \times \underline{\mathbf{v}}) \cdot \underline{\mathbf{w}} - (\nabla \times \underline{\mathbf{w}}) \cdot \underline{\mathbf{v}}$$

substituting $\underline{\mathbf{v}} = \nabla\phi$ and $\underline{\mathbf{w}} = \nabla\psi$ gives

$$\nabla \cdot (\nabla\phi \times \nabla\psi) = (\nabla \times \nabla\phi) \cdot \nabla\psi - (\nabla \times \nabla\psi) \cdot \nabla\phi = 0$$

because $\nabla \times (\nabla\phi) = 0$ for any ϕ (as shown in lecture).