PC10372, Mathematics 2
Example Sheet 6

1) Confirm the general identity,

$$\nabla \times (\nabla \times \underline{\mathbf{v}}) = \nabla (\nabla \cdot \underline{\mathbf{v}}) - \nabla^2 \underline{\mathbf{v}}$$

by evaluating each side explicitly when

$$\underline{\mathbf{v}} = \left(x^2 z \underline{\mathbf{i}} + y^2 x \underline{\mathbf{j}} + z^2 y \underline{\mathbf{k}}\right)/2$$

2) In free space, the electric field $\mathbf{E}$ and magnetic field $\mathbf{B}$ satisfy the Maxwell equations

$$\nabla \cdot \mathbf{E} = 0 \qquad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$
$$\nabla \cdot \mathbf{B} = 0 \qquad \nabla \times \mathbf{B} = \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}$$

where c is a constant.

Take the curl of each side of the two vector equations and use the identity in question 1 to show that

$$\nabla^2 \mathbf{E} = \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2}$$
$$\nabla^2 \mathbf{B} = \frac{1}{c^2}\frac{\partial^2 \mathbf{B}}{\partial t^2}$$

Substitute $\mathbf{E} = \underline{\mathbf{i}} \sin(kz - \omega t)$ and hence show that it is a possible solution for the electric field. Discuss the significance of this result.

3) Prove the following identity:

$$\nabla \cdot (\nabla \phi \times \nabla \psi) = 0$$

where ϕ and ψ are scalar fields.