PC10372, Mathematics 2
Example Sheet 5
Solutions

1) a) $\frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 3$
b) $\frac{\partial}{\partial x}(yz) + \frac{\partial}{\partial y}(xz) + \frac{\partial}{\partial z}(xy) = 0$
c) $\frac{\partial}{\partial x}(e^{x+y}) + \frac{\partial}{\partial y}(e^{4z}) + \frac{\partial}{\partial z}(e^{-3zx}) = e^{x+y} - 3xe^{-3zx}$
d) $\frac{\partial}{\partial x}(4) + \frac{\partial}{\partial y}(-1) + \frac{\partial}{\partial z}(3xz^2) = 6xz$
e)
$$\frac{\partial}{\partial x}\left(\frac{-xy}{(x^2+y^2)^{1/2}}\right) + \frac{\partial}{\partial y}\left(\frac{x^2}{(x^2+y^2)^{1/2}}\right) = -\frac{y}{(x^2+y^2)^{1/2}}$$

f) $\frac{\partial}{\partial x}(x\ \sin y) + \frac{\partial}{\partial y}(\cos y) + \frac{\partial}{\partial z}(xy) = \sin y - \sin y + 0 = 0$

2) a)
$$\begin{vmatrix} \underline{\mathbf{i}} & \underline{\mathbf{j}} & \underline{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z^2 & x^2 & -y^2 \end{vmatrix} = \underline{\mathbf{i}}(-2y) - \underline{\mathbf{j}}(-2z) + \underline{\mathbf{k}}(2x) = 2(-y\underline{\mathbf{i}} + z\underline{\mathbf{j}} + x\underline{\mathbf{k}})$$

b)
$$\begin{vmatrix} \underline{\mathbf{i}} & \underline{\mathbf{j}} & \underline{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 4xz & 0 & x^2 \end{vmatrix} = \underline{\mathbf{i}}(0) - \underline{\mathbf{j}}(2x - 4x) + \underline{\mathbf{k}}(0) = 2x\underline{\mathbf{j}}$$

c)
$$\begin{vmatrix} \underline{\mathbf{i}} & \underline{\mathbf{j}} & \underline{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^{-y} & e^{-z} & e^{-x} \end{vmatrix} = \underline{\mathbf{i}}(e^{-z}) - \underline{\mathbf{j}}(-e^{-x}) + \underline{\mathbf{k}}(e^{-y}) = e^{-z}\underline{\mathbf{i}} + e^{-x}\underline{\mathbf{j}} + e^{-y}\underline{\mathbf{k}}$$

d) Note that $(x^2+y^2+z^2)^{1/2} = r$ and $\frac{\partial r}{\partial x} = x/r$ etc., so

$$\begin{aligned} \begin{vmatrix} \underline{\mathbf{i}} & \underline{\mathbf{j}} & \underline{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{x}{r} & \frac{y}{r} & \frac{z}{r} \end{vmatrix} &= \underline{\mathbf{i}}\left(\frac{\partial}{\partial y}\frac{z}{r} - \frac{\partial}{\partial z}\frac{y}{r}\right) - \underline{\mathbf{j}}\left(\frac{\partial}{\partial x}\frac{z}{r} - \frac{\partial}{\partial z}\frac{x}{r}\right) + \underline{\mathbf{k}}\left(\frac{\partial}{\partial x}\frac{y}{r} - \frac{\partial}{\partial y}\frac{x}{r}\right) \\ &= \underline{\mathbf{i}}\left(\frac{-zy}{r^3} + \frac{zy}{r^3}\right) - \underline{\mathbf{j}}\left(\frac{-zx}{r^3} + \frac{zx}{r^3}\right) + \underline{\mathbf{k}}\left(\frac{-xy}{r^3} + \frac{xy}{r^3}\right) = 0 \end{aligned}$$

Notice that this field is just $\hat{\underline{\mathbf{r}}}$ and so is everywhere pointed away from the origin and so must have zero curl.

3) a) In spherical polar coordinates:

$$
\begin{aligned}
z = r\ \cos\theta, x &= r\ \sin\theta\cos\phi, y = r\ \sin\theta\sin\phi \\
2xz = 2r^2\ \sin\theta\ \cos\theta\ \cos\phi &= r^2\ \sin 2\theta\ \cos\phi \\
\nabla\left(2xz\right) &= \hat{\underline{\mathbf{r}}}\frac{\partial}{\partial r}\left(r^2\ \sin 2\theta\ \cos\phi\right) \\
&\quad +\hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial}{\partial\theta}\left(r^2\ \sin 2\theta\ \cos\phi\right) \\
&\quad +\hat{\boldsymbol{\phi}}\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\left(r^2\ \sin 2\theta\ \cos\phi\right) \\
&= \hat{\underline{\mathbf{r}}}2r\sin 2\theta\cos\phi + \hat{\boldsymbol{\theta}}2r\cos 2\theta\cos\phi - \hat{\boldsymbol{\phi}}\,2r\cos\theta\sin\phi
\end{aligned}
$$

where we have used $2\sin\theta\cos\theta = \sin 2\theta$ twice.

In cylindrical polar coordinates:

$$
\begin{aligned}
z = z, x &= r\ \cos\theta, y = r\ \sin\theta \\
\nabla(2xz) = \nabla(2r\cos\theta\, z) &= \hat{\underline{\mathbf{r}}}\frac{\partial}{\partial\theta}\left(2r\cos\theta\, z\right) + \hat{\boldsymbol{\theta}}\frac{\partial}{\partial r}\left(2r\cos\theta\, z\right) + \underline{\mathbf{k}}\frac{\partial}{\partial z}\left(2r\cos\theta\, z\right) \\
&= 2z\cos\theta\hat{\underline{\mathbf{r}}} - 2z\sin\theta\hat{\boldsymbol{\theta}} + 2r\cos\theta\underline{\mathbf{k}}
\end{aligned}
$$

4) a) Spherical polars

$$
\begin{aligned}
z &= r\cos\theta \\
\nabla z &= \frac{\partial}{\partial r}\left(r\cos\theta\right)\hat{\underline{\mathbf{r}}} + \frac{1}{r}\frac{\partial}{\partial\theta}\left(r\cos\theta\right)\underline{\hat{\boldsymbol{\theta}}} + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\left(r\cos\theta\right)\underline{\hat{\boldsymbol{\phi}}} \\
&= \cos\theta\hat{\underline{\mathbf{r}}} - \sin\theta\underline{\hat{\boldsymbol{\theta}}}
\end{aligned}
$$

and in Cartesian coordinates

$$
\begin{aligned}
\nabla z &= \frac{\partial}{\partial x}\left(z\right)\underline{\mathbf{i}}\frac{\partial}{\partial y}\left(z\right)\underline{\mathbf{j}} + \frac{\partial}{\partial z}\left(z\right)\underline{\mathbf{k}} \\
&= \underline{\mathbf{k}}
\end{aligned}
$$

These two results must represent the same result as the result can not depend on the coordinate system, so $\underline{\mathbf{k}} = \cos\theta\hat{\underline{\mathbf{r}}} - \sin\theta\underline{\hat{\boldsymbol{\theta}}}$.