PC10372, Mathematics 2
Example Sheet 4 Solutions

1) a) The product rule shows that

$$\frac{\partial}{\partial x}(fg) = f\frac{\partial g}{\partial x} + g\frac{\partial f}{\partial x}$$

and similarly for the partials w.r.t. y and z. So

$$\begin{aligned}\nabla(fg) &= f\left(\frac{\partial g}{\partial x}\mathbf{i} + \frac{\partial g}{\partial y}\mathbf{j} + \frac{\partial g}{\partial z}\mathbf{k}\right) + g\left(\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}\right)\\ &= f\nabla g + g\nabla f\end{aligned}$$

By the quotient rule

$$\frac{\partial}{\partial x}\left(\frac{f}{g}\right) = \frac{1}{g^2}\left(g\frac{\partial f}{\partial x} - f\frac{\partial g}{\partial x}\right)$$

and similarly for the partials w.r.t. y and z. So

$$\begin{aligned}\nabla\left(\frac{f}{g}\right) &= \frac{1}{g^2}g\left(\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}\right) - \frac{1}{g^2}f\left(\frac{\partial g}{\partial x}\mathbf{i} + \frac{\partial g}{\partial y}\mathbf{j} + \frac{\partial g}{\partial z}\mathbf{k}\right)\\ &= \frac{1}{g^2}(g\nabla f - f\nabla g)\end{aligned}$$

$$\begin{aligned}\frac{\partial}{\partial x}f^n = nf^{n-1}\frac{\partial f}{\partial x},\ \frac{\partial}{\partial y}f^n &= nf^{n-1}\frac{\partial f}{\partial y},\ \frac{\partial}{\partial z}f^n = nf^{n-1}\frac{\partial f}{\partial z}\\ \nabla f^n &= nf^{n-1}\left(\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\right)\\ &= nf^{n-1}\nabla f\end{aligned}$$

b) $f\nabla$ is a vector operator whereas ∇f is a the gradient of the field f and is a vector field.

2) a) $z = 2 - x - y$. Define $g(x, y, z) = 2 - x - y - z = 0$. The vector ∇g is normal to any surface of constant g. So

$$\begin{aligned}\nabla g &= \frac{\partial g}{\partial x}\mathbf{i} + \frac{\partial g}{\partial y}\mathbf{j} + \frac{\partial g}{\partial z}\mathbf{k}\\ &= -\mathbf{i} - \mathbf{j} - \mathbf{k}\\ \hat{\mathbf{n}} &= \ = \frac{-1}{3^{1/2}}(\mathbf{i} + \mathbf{j} + \mathbf{k})\end{aligned}$$

(b) Define $g(x, y, z) = z - (1 - x^2)^{1/2} = 0$,

$$\frac{\partial g}{\partial x} = \frac{x}{(1-x^2)^{1/2}},\ \frac{\partial g}{\partial y} = 0,\ \frac{\partial g}{\partial z} = 1$$

$$
\begin{aligned}
\nabla g &= \frac{x}{(1-x^2)^{1/2}}\mathbf{i}+\mathbf{k} \\
|\nabla g| &= \left(\frac{x^2}{(1-x^2)}+1\right)^{1/2} = \frac{1}{(1-x^2)^{1/2}} \\
\hat{\mathbf{n}} &= \frac{\nabla g}{|\nabla g|} = x\mathbf{i}+(1-x^2)^{1/2}\mathbf{k} \\
&= x\mathbf{i}+z\mathbf{k}
\end{aligned}
$$

This result can be seen by inspection: $g=0 \to z=(1-x^2)^{1/2}$ so $z^2+x^2=1$ which is a unit circle in the (x,z) plane. The normal to this circle is a radial unit vector $\mathbf{r}$ in the (x,z) plane, $\mathbf{r}=x\mathbf{i}+\mathbf{k}$.

(c) Define $g(x,y,z)=(x^2+y^2)^{1/2}-z=0$

$$
\begin{aligned}
\frac{\partial g}{\partial x}=\frac{x}{(x^2+y^2)^{1/2}},\ \frac{\partial g}{\partial y} &= \frac{y}{(x^2+y^2)^{1/2}},\ \frac{\partial g}{\partial z}=-1 \\
\nabla g &= \frac{x}{(x^2+y^2)^{1/2}}\mathbf{i}+\frac{y}{(x^2+y^2)^{1/2}}\mathbf{j}-\mathbf{k} \\
|\nabla g| &= \frac{x^2}{(x^2+y^2)}+\frac{y^2}{(x^2+y^2)}+1 = \sqrt{2} \\
\hat{\mathbf{n}} &= \frac{\nabla g}{|\nabla g|} = \frac{1}{\sqrt{2}}\left(\frac{x}{(x^2+y^2)^{1/2}}\mathbf{i}+\frac{y}{(x^2+y^2)^{1/2}}\mathbf{j}-\mathbf{k}\right) \\
&= \frac{1}{\sqrt{2}}\left(\frac{x}{z}\mathbf{i}+\frac{y}{z}\mathbf{j}-\mathbf{k}\right)
\end{aligned}
$$

(d) Define $g(x,y)=x^2+y^2-z=0$

$$
\begin{aligned}
\frac{\partial g}{\partial x}=2x,\ \frac{\partial g}{\partial y} &= 2y,\ \frac{\partial g}{\partial z}=-1 \\
\nabla g &= 2x\mathbf{i}+2y\mathbf{j}-\mathbf{k} \\
|\nabla g| &= \sqrt{4x^2+4y^2+1}=\sqrt{4z+1} \\
\hat{\mathbf{n}} &= \frac{\nabla g}{|\nabla g|} = \frac{1}{\sqrt{4z+1}}\left(2x\mathbf{i}+2y\mathbf{j}-\mathbf{k}\right)
\end{aligned}
$$

3) The inverse rotation is just a rotation by angle $-\theta$, so

$$
\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix}
$$

Giving

$$
\begin{aligned}
x &= \cos\theta x' - \sin\theta y' \\
y &= \sin\theta x' + \cos\theta y'
\end{aligned}
$$

The chain rule gives

$$\begin{aligned}
\frac{\partial}{\partial x'} &= \frac{\partial x}{\partial x'}\frac{\partial}{\partial x} + \frac{\partial y}{\partial x'}\frac{\partial}{\partial y} \\
&= \cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y} \\
\frac{\partial}{\partial y'} &= \frac{\partial x}{\partial y'}\frac{\partial}{\partial x} + \frac{\partial y}{\partial y'}\frac{\partial}{\partial y} \\
&= -\sin\theta\frac{\partial}{\partial x} + \cos\theta\frac{\partial}{\partial y}
\end{aligned}$$

Giving

$$\begin{pmatrix} \frac{\partial}{\partial x'} \\ \frac{\partial}{\partial y'} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix}$$

showing that the operator transforms in the same way as $\begin{pmatrix} x \\ y \end{pmatrix}$ which demonstrates that ∇f is indeed a vector quantity.