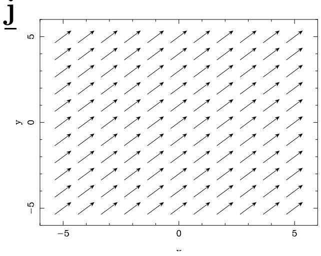
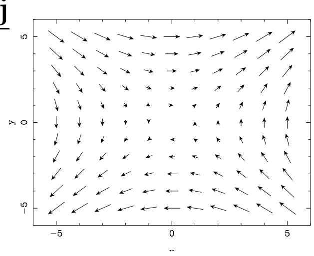
PC10372, Mathematics 2
Example Sheet 3 Solutions

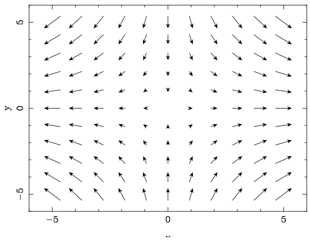
1) i) $\nabla\phi = \nabla(x+y) = \underline{\mathbf{i}} + \underline{\mathbf{j}}$



ii) $\nabla\phi = \nabla(xy) = y\underline{\mathbf{i}} + x\underline{\mathbf{j}}$



iii) $\nabla\phi = \nabla(x^2 - y^2) = 2x\underline{\mathbf{i}} - 2y\underline{\mathbf{j}}$



2)

$$r = \sqrt{x^2+y^2+z^2} \quad \frac{\partial r}{\partial x} = \frac{2x}{2\sqrt{x^2+y^2+z^2}} = \frac{x}{r}$$

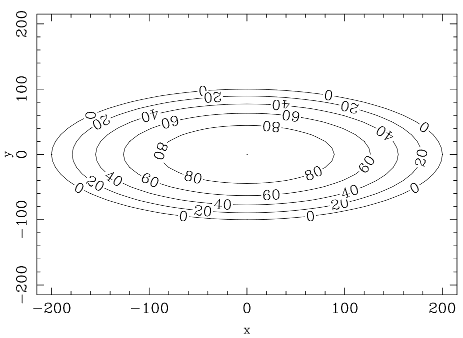
$$\nabla r = \frac{x}{r}\underline{\mathbf{i}} + \frac{y}{r}\underline{\mathbf{j}} + \frac{z}{r}\underline{\mathbf{k}} == \frac{1}{r}\left(x\underline{\mathbf{i}} + y\underline{\mathbf{j}} + z\underline{\mathbf{k}}\right) = \frac{1}{r}\underline{\mathbf{r}} = \underline{\hat{\mathbf{r}}}$$

$$\nabla\left(\frac{1}{r}\right) = -\frac{1}{r^2}\frac{\partial r}{\partial x}\underline{\mathbf{i}} + -\frac{1}{r^2}\frac{\partial r}{\partial y}\underline{\mathbf{j}} - \frac{1}{r^2}\frac{\partial r}{\partial z}\underline{\mathbf{k}} == -\frac{1}{r^3}\underline{\mathbf{r}} = -\frac{\underline{\hat{\mathbf{r}}}}{r^2}$$

3) a)

$$h^2 = h_0^2 - \frac{x^2}{4} - y^2, \text{ so } \frac{x^2}{4} + y^2 = h^2 - h_0^2$$
$$\frac{x^2}{4(h^2-h_0^2)} + \frac{y^2}{h^2-h_0^2} = 1$$

- contours of constant height (h constant) are ellipses with semi-major axis $2\sqrt{h^a2 - h_0^2}$ and semi-minor axis $\sqrt{h^2-h_0^2}$.



b)

$$
\begin{aligned}
h &= \left(h_0^2 - \frac{x^2}{4} - y^2\right)^{1/2} \\
\frac{\partial h}{\partial x} &= \frac{-x/2}{2\sqrt{h_0^2 - \frac{x^2}{4} - y^2}} = \frac{-x}{4h}
\end{aligned}
$$

Similarly $\frac{\partial h}{\partial y} = -y/h$ so

$$
\nabla h = -\frac{x}{4h}\underline{\mathbf{i}} - \frac{y}{h}\underline{\mathbf{j}}
$$

To find the gradient in the direction $\underline{\mathbf{a}} = 2\underline{\mathbf{i}} + \underline{\mathbf{j}}$ take the scalar product with the unit vector in this direction, $\hat{\underline{\mathbf{a}}} = \underline{\mathbf{a}}\sqrt{5}$

$$
\nabla h \cdot \frac{(2\underline{\mathbf{i}} + \underline{\mathbf{j}})}{\sqrt{5}} = -\frac{2x}{4h\sqrt{5}} - \frac{y}{h\sqrt{5}} = -\frac{1}{h\sqrt{5}}\left(\frac{x}{2} + y\right)
$$

Evaluating at (-50m, -30m),

$$
\nabla h \cdot \frac{(2\underline{\mathbf{i}} + \underline{\mathbf{j}})}{\sqrt{5}} = -\frac{1}{h\sqrt{5}}(-25 + 30) = -\frac{\sqrt{5}}{h}
$$

which is less than zero and so walking downhill.

c) Direction of steepest ascent is $\underline{\mathbf{n}}$ where

$$
\underline{\mathbf{n}} = \frac{\nabla h}{|\nabla h|} = \frac{1}{|\nabla h|}\left(-\frac{x}{4h}\underline{\mathbf{i}} - \frac{y}{h}\underline{\mathbf{j}}\right)
$$

On the steepest path $\underline{\mathbf{n}}$ is tangent to the path at every point and the angle θ between $\underline{\mathbf{n}}$ and the x axis is given by

$$
\tan\theta = \frac{-y/h}{-x/4h} = \frac{4y}{x}
$$

But $\tan\theta = \frac{dx}{dy}$, the slope of the path, so

$$
\begin{aligned}
\frac{dx}{dy} &= \frac{4y}{x} \\
\int \frac{dy}{y} &= \int 4\frac{dx}{x} \\
\ln y &= 4\ln x + C \\
y &= Nx^4
\end{aligned}
$$

as required.

To find the value of the constant N, substitute the starting position, $(-120, -80)$ so $-80 = N(-120)^4$ giving $N = -3.86 \times 10^{-7}$ m^{-3}.